Arithmetic Sequences Lesson #2:
Arithmetic Sequences

$pg.517$

Arithmetic Sequence

An **arithmetic sequence** is a sequence in which each term is formed from the preceding term by *adding* a constant (positive or negative).

$t_1 \ t_2 \ t_3 \ t_4$

Complete the following for the sequence $7, 10, 13, 16, \ldots$.

• Each term is determined by adding __3__ to the previous term.

• Calculate the differences: $t_2 - t_1 = $ __3__ $t_3 - t_2 = $ __3__ $t_4 - t_3 = $ __3__

   $10-7$ $13-10$ $16-13$

Notice that there is a common difference between successive terms.

The common difference in this example is __3__.

Finding a Common Difference

To find a common difference in an arithmetic sequence, we can subtract any term from the term after it.

   For example $t_2 - t_1 = $ common difference, or
   $t_5 - t_4 = $ common difference, etc.

$$\text{common difference} = t_n - t_{n-1}$$

Class Ex. #1 Consider the sequence $16, 13, 10, 7, \ldots$.

The **common difference** in the sequence is __-3__ .

$t_2 - t_1$ $t_3 - t_2$
$= 13-16$ $= 10-13$
$= -3$ $= -3$

negative ⟹ decreasing in value

Class Ex. #2 For each of the following :
• Determine which sequences are arithmetic. — a common difference must exist
• Find the common difference for those sequences which are arithmetic .

a) $2, 4, 6, 8, \ldots$ b) $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16} \ldots$ c) $-10, -4, 2, 8, \ldots$ d) $4, 8, 16, 32, \ldots$

arithmetic $\times\frac{1}{2} \ \times\frac{1}{2} \ \times\frac{1}{2}$ arithmetic $\times2 \ \times2 \ \times2$

Common diff = 2   not arithmetic   Common diff = 6   not arithmetic
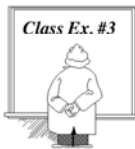
Note In an arithmetic sequence we often use the following terminology.

The **first term** in an arithmetic sequence is represented by t_1, or a, and the **common difference** is represented by d.

Class Ex. #3   State the values of a and d in the following sequences:

1st term   *Common difference*

i) $-8, 2, 12, 22, ...$

$a = -8$
$d = 10$

ii) $15, 10, 5, 0 ....$

$a = 15$
$d = -5$

Investigation | *Investigating the Formula for the General Term*
of an Arithmetic Sequence

Consider the sequence $2, 12, 22, 32, 42 ... .$

a) State the following

$t_1 = 2$ $t_2 = 12$ $t_3 = 22$ $t_4 = 32$ $t_5 = 42$ $a = 2$ $d = 10$

b) Complete the following pattern which describes each term in the sequence in terms of the first term, a, and the common difference, d.

$t_1 = 2$	$t_1 = a$
$t_2 = 2 + 1(10) = 12$	$t_2 = a + (1)d$
$t_3 = 2 + 2(10) = 22$	$t_3 = a + 2d$
$t_4 = 2 + 3(10) = 32$	$t_4 = a + 3d$
$t_5 = 2 + 4(10) = 42$	$t_5 = a + 4d$
$t_{30} = 2 + 29(10) = 292$	$t_{30} = a + 29d$
$t_n = 2 + (n-1)10$	$\boxed{t_n = a + (n-1)d}$

The Formula for the General Term of an Arithmetic Sequence

The formula for the general term of an arithmetic sequence is

$$\boxed{t_n = t_1 + (n-1)d}$$ **or** $$\boxed{t_n = a + (n-1)d}$$

where, t_n is the general term of the arithmetic sequence
$a = t_1$, is the first term
d is the common difference
n is the position of the term in the sequence

Note   The general arithmetic sequence is $a, a + d, a + 2d, a + 3d, ... a + (n-1)d.$

Class Ex. #4 Consider the arithmetic sequence $-6, -1, 4, 9, \ldots$ t_n
a) Determine the formula for the general term of the sequence.

$a = -6$
$d = 5$

$$t_n = a + (n-1)d$$
$$t_n = -6 + (n-1)(5)$$
$$t_n = -6 + 5n - 5$$
$$t_n = 5n - 11$$

b) Determine the value of the twelfth term of the sequence. $n = 12$

$$t_{12} = 5(12) - 11$$
$$t_{12} = 60 - 11$$
$$t_{12} = 49$$

Class Ex. #5 Find the number of terms in the arithmetic sequence $3, -1, -5, \ldots, -117$. $n = ?$

$a = 3$
$d = -4$
$t_n = -117$
$n = ?$

$$t_n = a + (n-1)d$$
$$-117 = 3 + (n-1)(-4)$$
$$-117 = 3 - 4n + 4$$
$$1 + \quad 7 = 7 - 4n$$
$$\frac{-124}{-4} = \frac{-4n}{-4}$$
$$n = 31$$

There are 31 terms in the sequence.

> **Complete Assignment Questions #1 - #9**

> ## Arithmetic Means

The terms placed between two non-consecutive terms of an arithmetic sequence are called **arithmetic means**. For example, in the sequence $5, 10, 15, 20$, the numbers 10 and 15 are arithmetic means between 5 and 20. In order to determine arithmetic means between two given terms, it is helpful to think of the two given terms as the **first** and **last** terms of a sequence.

Class Ex. #6 Place three arithmetic means between -4 and 8.

$$-4, \underset{+d}{\overset{+3}{-1}}, \underset{+d}{\overset{+3}{2}}, \underset{+d}{\overset{+3}{5}}, \underset{+d}{8}$$

The 3 arithmetic means are -1, 2 and 5.

$$-4 + 4d = 8$$
$$\frac{4d}{4} = \frac{12}{4}$$
$$d = 3$$

Solving Sequence Problems Where Both "a" and "d" are Unknown

Class Ex. #7

Consider the sequence $\overset{t_1}{x+2},\ \overset{t_2}{3x-1},\ \overset{t_3}{2x+1}$.

a) Determine the value of x such that $x+2$, $3x-1$, and $2x+1$ form an arithmetic sequence.

$$d = t_2 - t_1 \qquad \text{and} \qquad d = t_3 - t_2$$
$$t_2 - t_1 = t_3 - t_2$$
$$(3x-1) - (x+2) = (2x+1) - (3x-1)$$
$$3x - 1 - x - 2 = 2x + 1 - 3x + 1$$
$$2x - 3 = -x + 2$$

$$\frac{3x}{3} = \frac{5}{3}$$
$$x = \frac{5}{3}$$

b) Determine the numerical value of the three terms.

$$t_1 = x + 2 \qquad\qquad t_2 = 3x - 1 \qquad\qquad t_3 = 2x + 1$$
$$= \frac{5}{3} + \frac{6}{3} \qquad\qquad = 3\left(\frac{5}{3}\right) - 1 \qquad\qquad = 2\left(\frac{5}{3}\right) + 1$$
$$= \frac{11}{3} \qquad\qquad = 4 \qquad\qquad = \frac{10}{3} + \frac{3}{3} = \frac{13}{3}$$

$$\boxed{\frac{11}{3},\ 4,\ \frac{13}{3}}$$

The next two class examples show two different ways of solving the same problem. Class Example #8 uses arithmetic means, and Class Example #9 uses a system of linear equations.

Class Ex. #8

The third and eighth terms of an arithmetic sequence are 12 and −18, respectively.

a) Use arithmetic means to determine the fifth term of the sequence.

$$24,\ 18,\ 12,\ 6,\ \boxed{0},\ -6,\ -12,\ -18$$

(with $+d$ between each term)

$$\boxed{t_5 = 0}$$

$$12 + 5d = -18$$
$$-12 \qquad\qquad -12$$
$$5d = -30$$
$$d = -6$$

b) State the first term, a, and the common difference, d, of the sequence.
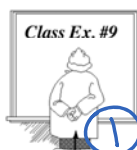
$$a = 24 \qquad d = -6$$

c) Complete the following: $\dfrac{t_8 - t_3}{8 - 3} = \dfrac{\quad\quad}{\quad\quad} =$

d) Write t_3 and t_8 in terms of a and d and prove that $\dfrac{t_8 - t_3}{8 - 3} = d$.

e) Suggest a formula for finding the common difference of a sequence if you are given the value of the pth term and the qth term.

$t_n = a + (n-1)d$ a & d are unknown

Class Ex. #9

The third and eighth terms of an arithmetic sequence are 12 and –18, respectively. Use a system of linear equations to determine the values of the first term and the common difference. Hence, determine the fifth term of the sequence.

① $t_3 = 12$ $t_8 = -18$

$t_3 = a + (3-1)d$ $t_8 = a + (8-1)d$

$a + 2d = 12$ $a + 7d = -18$

② Solve by elimination

$a + 7d = -18$

$- (a + 2d = 12)$

$5d = -30$

$d = -6$

③ $a + 2d = 12$

$a + 2(-6) = 12$

$a - 12 = 12$

$a = 24$

④ Find t_5

$t_5 = 24 + (5-1)(-6)$

$= 24 + 4(-6)$

$= 24 - 24$

$= 0$

Complete Assignment Questions #10 - #16

Assignment

1. For the following arithmetic sequences:
   i) Determine the common difference.   ii) Find the next three terms of the sequence.

   a) $8, 14, 20, ...$ b) $-5, 7, 19, ...$ c) $70, 53, 36, 19, ...$

   d) $7.1, 4.2, 1.3, ...$ e) $\dfrac{2}{3}, \dfrac{1}{15}, -\dfrac{8}{15}, ...$ f) $-2x + 3y, -5x + y, -8x - y, ...$

2. In each of the following sequences, the value of one term is given. Write the missing terms of the sequence if the common difference is as indicated.

   a) ____, ____, 0, ____, ____ : $d = 3$ b) ____, ____, ____, –3, ____ : $d = -7$

   c) ____, ____, 1, ____, ____ : $d = -2$ d) ____, ____, ____, ____, 15: $d = 2.5$