Factoring Polynomial Expressions Lesson #1:
Common Factors

Overview of Unit

In this unit, we introduce the process of factoring. This includes factoring by removing a common factor, factoring a trinomial, and factoring a difference of squares. These techniques are illustrated concretely, pictorially, and symbolically. We express a polynomial as a product of its factors and include, for enrichment, polynomial equation solving.

Expanding and Factoring

In the previous unit, we were concerned with multiplying polynomial expressions.
In particular we multiplied

i) a monomial by a polynomial e.g. $2x(x+5) = $ $2x^2 + 10x$

ii) a binomial by a binomial to form a trinomial e.g. $(x+1)(x+3) = $ x^2+4x+3

FOIL $\longrightarrow$ $(x)(x) + (x)(3) + (1)(x) + (1)(3)$

iii) a binomial by a binomial to form a binomial e.g. $(x-5)(x+5) = $ x^2-25

expanding a difference of squares.

In these examples, we have **expanded** a product of polynomials to form a sum or difference of monomials.

In this unit, we are concerned with the opposite process. We want to write a sum or difference of monomials as a product of polynomials. This process is called **factoring.**

We will be studying the following three major types of factoring.

Complete the following using the results obtained above.

i) factoring by removing a common factor e.g. $2x^2 + 10x = $ $2x(x+5)$

ii) factoring a trinomial. e.g. $x^2 + 4x + 3 = $ $(x+1)(x+3)$

iii) factoring a difference of squares e.g. $x^2 - 25 = $ $(x-5)(x+5)$

Greatest Common Factor

In the lesson "Applications of Prime Factors" page 9, we met the concept of the greatest common factor of whole numbers.

The GCF of 48 and 72 was found by using prime factorization.

$48 = 2 \times 2 \times 2 \times 2 \times 3$ and $72 = 2 \times 2 \times 2 \times 3 \times 3$

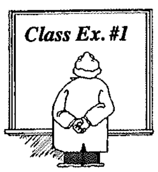
To determine the greatest common factor of 48 and 72, we found the product of each prime factor (including repeats) which is common to each prime factorization.

GCF of 48 and 72 is $2 \times 2 \times 2 \times 3 =$ 24.

The same process can be used to determine the greatest common factor of two monomials like $6a^3$ and $9a^2b$.

$6a^3 = 2 \times 3 \times a \times a \times a$ and $9a^2b = 3 \times 3 \times a \times a \times b$

GCF of $6a^3$ and $9a^2b$ is $3 \times a \times a = 3a^2$.



Write the prime factorization of $8x^2y^2$ and $20xy^3$ and determine the greatest common factor of $8x^2y^2$ and $20xy^3$.

8 → 2, 4 → 2, 2

Recall: use a divison table or factor tree to complete prime Factorization!

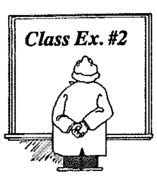
2 | 20
2 | 10
5 | 5
1 ← STOP

$8x^2y^2 = 2 \times 2 \times 2 \times x \times x \times y \times y$

$20xy^3 = 2 \times 2 \times 5 \times x \times y \times y \times y$

GCF $= 2 \times 2 \times x \times y \times y = 4xy^2$

- The greatest common factor of two simple monomials can be determined by inspection, by taking the GCF of any numerical coefficients and multiplying by each common variable to the lowest common exponent.
  The greatest common factor of $10p^3q$ and $15p^2q^2$ is determined by multiplying 5 by p^2 by q, i.e. $5p^2q$.
- If all the monomials are negative, the GCF is usually considered to be negative (see example d) below).



In each case, state the greatest common factor of the following sets of monomials.

a) $12ab,\ 15a^2b^3$ — 3ab

12 → 2, 6 → 2, 3; 15 → 3, 5

b) $18x^4y^2,\ -24x^3y^5$ — $6x^3y^2$

c) $a^3bc^2,\ 2ac^7$ — ac^2

d) $-40a^3b,\ -20a^2b^3,\ -10a^2b^2$ — $-10a^2b$

Complete Assignment Question #1 - #3

Factoring a Polynomial by Removing the Greatest Common Factor

Factoring is a process in which a sum or difference of terms is expressed as a product of factors.

A polynomial like $8x^2y^2 + 20xy^3$ can be factored by removing (or taking out, or dividing out) the greatest common factor from each term.

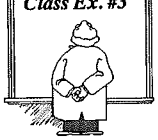
We know that

$4xy^2(2x + 5y)$ can be <u>expanded</u> to give $8x^2y^2 + 20xy^3$.

It follows that

$8x^2y^2 + 20xy^3$ can be <u>factored</u> to give $4xy^2(2x + 5y)$.

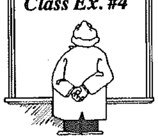
In this case, the greatest common factor $4xy^2$ has been removed from each term.


Class Ex. #3

In each case, complete the factoring.

a) $21x + 14y = \underline{\ 7\ }(3x + 2y)$ must be the GCF

b) $5x^4 + 15x^3 + 5x^2 = \underline{\ 5x^2\ }(x^2 + 3x + 1)$


Class Ex. #4

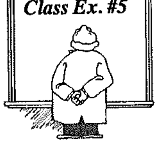
In each case, the greatest common factor has been removed. Complete the factoring.

a) $5a^2 + 25a = 5a(a + 5)$ what multiplied by 5a is 25a?

b) $18p - 16q = 2(9p - 8q)$

c) $-4mn - 6m^2 = -2m(2n + 3m)$

d) $18x^2y^2 - 45xy + 9x = 9x(2xy^2 - 5y + 1)$


Class Ex. #5

Factor each polynomial by removing the greatest common factor.

lowest common exponent

a) $20x^1 - 6$

$= 2(10x - 3)$

b) $16x^4 + 4x^2$

$= 4x^2(4x + 1)$

c) $10a^3b^2 + 8ab^3 + 2ab^4$

$= 2ab^2(5a^2 + 4b + b^2)$

d) $12p^3 - 6p^2 + 15p$

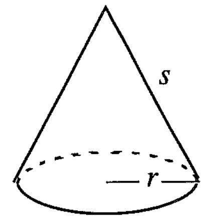
$= 3p(4p^2 - 2p + 5)$

e) $25xy^2z^3 - 20x^2y^4z^2 + 30x^4y^2z^5$

$= 5xy^2z^2(5z - 4xy^2 + 6x^3z^3)$

Class Ex. #6

The surface area of a cone is given by the formula $A = \pi r^2 + \pi rs$, where r is the radius of the base of the cone and s is the slant height.



i) Determine the surface area of a cone, to the nearest 0.01 cm^2, which has slant height 7.40 cm and base radius 2.60 cm.

$A = \pi(2.60)^2 + \pi(2.60)(7.40)$
$= 81.68 \text{ cm}^2$

ii) Write the formula for A in factored form.

$A = \pi r(r + s)$

iii) Calculate the surface area of the cone, to the nearest 0.01 cm^2, using the factored form of A.

$A = \pi(2.60)(2.60 + 7.40)$
$= 81.68 \text{ cm}^2$

iv) Which method i) or iii) is simpler to use? The factored form is simpler.

Complete Assignment Questions #4 - #13

Assignment

$2\underline{|12}$ $2\underline{|6}$ $3\underline{|3}$ 1 ← STOP

$2\underline{|30}$ $3\underline{|15}$ $5\underline{|5}$ 1 ← STOP

1. Write the prime factorization of $12a^3$ and $30a^2$ and determine the greatest common factor of $12a^3$ and $30a^2$.

$12a^3 = 2 \times 2 \times 3 \times a \times a \times a$
$30a^2 = 2 \times 3 \times 5 \times a \times a$

GCF $= 2 \times 3 \times a \times a = 6a^2$

10 → 2, 5   25 → 5, 5

Recall: Only stop factoring when all factors are prime.

2. Write the prime factorization of $10xy^4$ and $25x^2y^3$ and determine the greatest common factor of $10xy^4$ and $25x^2y^3$.

$10xy^4 = 2 \times 5 \times x \times y \times y \times y \times y$
$25x^2y^3 = 5 \times 5 \times x \times x \times y \times y \times y$

GCF $= 5 \times x \times y \times y \times y = 5xy^3$

3. In each case, state the greatest common factor of the following sets of monomials.

→ lowest common exponents

a) $7m^1$, $14m^1$ $7m$ **b)** $6x^2$, $9x^1$ $3x$ **c)** b^1c^2, b^1c^7 bc^2

d) a^1b^1, a^2b^2 ab **e)** $4x^4$, $8x^3$ $4x^3$ **f)** $3xyz$, $9rst$, $12def$ 3

Remember: lowest common exponents.

g) $-8pq^3,\ 18p^2q$ h) $-10x^5z^6,\ -15x^5z^4$ i) $8ab^2,\ 9ab,\ 6a^2b$

$2pq$ $-5x^5z^4$ ab

j) $10xy,\ 16xz,\ 20xyz$ k) $-2x^3y,\ -4x^3y^4,\ -4x^2y^4$ l) $-28pqr^3,\ -56p^2q,\ -64q^2r$

$2x$ $-2x^2y$ $-4q$

4. Complete the factoring in each case.

Recall: the factor that is common to all terms must be the GCF.

a) $12a + 24b = \underline{12}\ (a + 2b)$ b) $4p^2 - 7p = \underline{p}\ (4p - 7)$

c) $2xy + 3xz = \underline{x}\ (2y + 3z)$ d) $5x^2 + 10x + 15 = \underline{5}\ (x^2 + 2x + 3)$

e) $6cde - 4cd = \underline{2cd}\ (3e - 2)$ f) $3y^3 - 9y = \underline{3y}\ (y^2 - 3)$

5. In each case, the greatest common factor has been removed. Complete the factoring.

Check work.

	$3x^2$
$2x$	$6x^3$
-3	$-9x^2$

✓✓

a) $3a^2 + 15a = 3a(a + 5)$ b) $20p - 10q = 10(2p - q)$

c) $6x^3 - 9x^2 = 3x^2(2x - 3)$ d) $4a^2b + 8a^3b^2 = 4a^2b(1 + 2ab)$

e) $-15x^2y - 10x^2y^2 = -5x^2y(3 + 2y)$ f) $16xm^2n^3 - 12mn^2 - 4mn = 4mn(4xmn^2 - 3n - 1)$

6. Factor the following polynomials by removing the greatest common factor.

a) $6m + 6n$ b) $7xy^2 + 49$ c) $15pq - 5$ d) $8c + 12d$

$= 6(m + n)$ $= 7(xy^2 + 7)$ $= 5(3pq - 1)$ $= 4(2c + 3d)$

Check work!

	y
x	xy
1	y

✓✓

e) $xy + y$ f) $6x^2 - 9x$ g) $9ab - 12ac$ h) $48y^2 - 72y^5$ lowest common exponent

$= y(x + 1)$ $= 3x(2x - 3)$ $= 3a(3b - 4c)$ $= 24y^2(2 - 3y^3)$

7. Factor the following polynomials

a) $12x - 8y + 16z$ b) $9pq + 6pr - 15p$ c) $t^3 + t^2 + t$ lowest common exponent.

$= 4(3x - 2y + 4z)$ $= 3p(3q + 2r - 5)$ $= t(t^2 + t + 1)$

d) $5x^2 - 10xy - 20xz$ e) $4abc - 2abd + 8abe$ f) $14a^2b^2 + 21a^3b^2 - 35a^2b^3$

$= 5x(x - 2y - 4z)$ $= 2ab(2c - d - 4e)$ $= 7a^2b^2(2 + 3a - 5b)$

14 → 2, 7
21 → 3, 7
35 → 5, 7

Ensure that your common factor is the GCF by factoring. Use a factor tree or divison table if needed. Otherwise, you may not completely factor the given polynomial.

8. In each of the following:
i) simplify the expression by combining like terms.
ii) factor the resulting polynomial.

a) $5x^2 - 2x + 7 - 2x^2 + 8x - 7$

$= 3x^2 + 6x$ ← lowest common exponent.

$= 3x(x+2)$

b) $6 - 2y + 5y^2 - 10y + 3y^2 - 12$

$= 8y^2 - 12y - 6$

$= 2(4y^2 - 6y - 3)$

c) $xy^3 - 2x^3y + 6x^2y^2 - 5xy^3 + 8x^3y$

$= 6x^3y + 6x^2y^2 - 4xy^3$

$= 2xy(3x^2 + 3xy - 2y^2)$

Problem: We must expand first before we can simplify by combining like terms.

d) $2(x^3 - 3x) - 4x(x - 6) + 5x^2(x - 2) - 4x$

$= 2x^3 - 6x - 4x^2 + 24x + 5x^3 - 10x^2 - 4x$

$= 7x^3 - 14x^2 + 14x$

$= 7x(x^2 - 2x + 2)$

9. Factor the following polynomials. Expand your answer to verify the factoring.

a) $24x^3 - 60x^2$

$= 12x^2(2x-5)$

	$12x^2$
$2x$	$24x^3$
-5	$-60x^2$

$24x^3 - 60x^2$

b) $-8p^3 - 32p^2 - 8p$

$= -8p(p^2 + 4p + 1)$

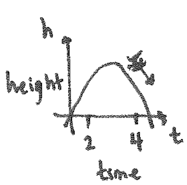
Expand $-8p(p^2+4p+1) = -8p^3 - 32p^2 - 8p$ to verify using the distributive property

10. An archer standing on the ground fires an arrow vertically upward into the air at a speed of 30 m/s.
The height (h metres) of the arrow above the ground after t seconds can be approximated by the formula $h = 30t - 5t^2$.

a) Write h in factored form. $h = 5t(6-t)$

b) Use the factored form of h to calculate the height for each of the times in the table. Record your answer in the table.

Time (t seconds)	0	1	2	3	4	5	6
Height (h metres)	0	25	40	45	40	25	0



c) Explain why the height of the arrow after two seconds is the same as the height of the arrow after four seconds.

At 2 seconds the arrow is on the way up and at 4 seconds it is on the way down.

d) Calculate h when $t = 7$. Explain why this has no meaning in the context of the question.

$h = 5(7)(6-7) = -35$

The arrow has already hit the ground at $t=6$. It does not travel 35m below the ground.

Multiple Choice **11.** One factor of $9x^4 - 6x^3 + 3x^2$ is

A. $9x^4$
B. $3x^2 - 2x$
C. $3x^2 - 6x + 3$
D. $3x^2 - 2x + 1$

GCF
$9x^4 = 3 \times 3 \times x \times x \times x \times x$
$6x^3 = 2 \times 3 \times x \times x \times x$
$3x^2 = 3 \times x \times x$

$3x^2(3x^2 - 2x + 1)$

Numerical Response **12.** When $x^4y^3 - x^2y^3 + x^6y$ is factored, the greatest common factor has degree A and the remaining trinomial factor has degree B. The value of $A + 2B$ is _____.

(Record your answer in the numerical response box from left to right)

1	1		

$x^2y(x^2y^2 - y^2 + x^4)$

degree of common factor $A = 2+1 = 3$
degree of remaining trinomial factor $B = 4$

$A + 2B = (3) + 2(4) = 11$ ← notice that brackets were used to show substitution.

13. When the greatest common factor is removed from the binomial $98x^2 - 28x$, the binomial can be written in the form $ax(bx + c)$. The value of $a + b + c$ is _____.

(Record your answer in the numerical response box from left to right)

1	9		

2 | 98
7 | 49
7 | 7
1 ← STOP

2 | 28
2 | 14
7 | 7
1 ← STOP

Now: $98x^2 = 2 \times 7 \times 7 \times x \times x$
$28x = 2 \times 2 \times 7 \times x$

$GCF = 2 \times 7 \times x = 14x$

Thus: $14x(7x - 2)$
$\downarrow$ a, $\downarrow$ b, $\downarrow$ c

So: $a = 14$
$b = 7$
$c = -2$

Therefore: $a + b + c = (14) + (7) + (-2)$
$= 19$

Answer Key

1. $12a^3 = 2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot a$ $30a^2 = 2 \cdot 3 \cdot 5 \cdot a \cdot a$ $GCF = 2 \cdot 3 \cdot a \cdot a = 6a^2$

2. $10xy^4 = 2 \cdot 5 \cdot x \cdot y \cdot y \cdot y \cdot y$ $25x^2y^3 = 5 \cdot 5 \cdot x \cdot x \cdot y \cdot y \cdot y$ $GCF = 5 \cdot x \cdot y \cdot y \cdot y = 5xy^3$

3. **a)** $7m$ **b)** $3x$ **c)** bc^2 **d)** ab **e)** $4x^3$ **f)** 3 **g)** $2pq$ **h)** $-5x^5z^4$ **i)** ab **j)** $2x$ **k)** $-2x^2y$ **l)** $-4q$

4. **a)** 12 **b)** p **c)** x **d)** 5 **e)** $2cd$ **f)** $3y$

5. **a)** $a+5$ **b)** $2p-q$ **c)** $2x-3$ **d)** $1+2ab$ **e)** $3+2y$ **f)** $4xmn^2 - 3n - 1$

6. **a)** $6(m+n)$ **b)** $7(xy^2+7)$ **c)** $5(3pq-1)$ **d)** $4(2c+3d)$
e) $y(x+1)$ **f)** $3x(2x-3)$ **g)** $3a(3b-4c)$ **h)** $24y^2(2-3y^3)$

7. **a)** $4(3x-2y+4z)$ **b)** $3p(3q+2r-5)$ **c)** $t(t^2+t+1)$ **d)** $5x(x-2y-4z)$
e) $2ab(2c-d+4e)$ **f)** $7a^2b^2(2+3a-5b)$

8. **a)** $3x^2+6x = 3x(x+2)$ **b)** $8y^2-12y-6 = 2(4y^2-6y-3)$
c) $6x^3y+6x^2y^2-4xy^3 = 2xy(3x^2+3xy-2y^2)$ **d)** $7x^3-14x^2+14x = 7x(x^2-2x+2)$

9. **a)** $12x^2(2x-5)$ **b)** $-8p(p^2+4p+1)$

10. **a)** $h = 5t(6-t)$ **b)** 0, 25, 40, 45, 40, 25, 0
c) At 2 sec. the arrow is on the way up and at 4 sec. the arrow is on the way down.
d) $h = -35$. The arrow has already hit the ground at $t = 6$. It does not travel 35m below the ground.

11. D

12.

1	1		

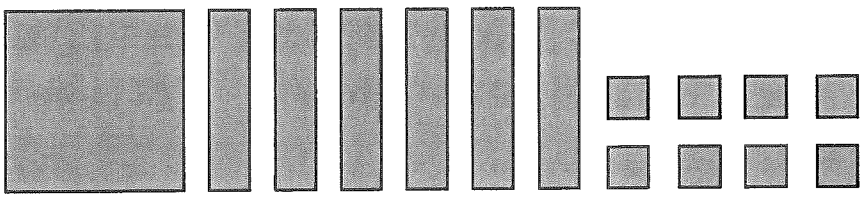
13.

1	9		

Factoring Polynomial Expressions Lesson #2:
Factoring Trinomials of the Form $x^2 + bx + c$ - Part One

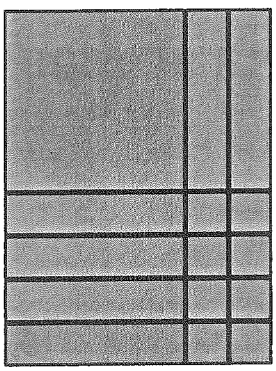
Factoring Trinomials using Algebra Tiles

Consider the algebra tile diagram shown.



- Write the polynomial expression which is represented by the algebra tiles.

- The algebra tiles can be rearranged into a rectangular form as shown below.



i) Write an expression for the length of the rectangle.

$x + 2$

ii) Write an expression for the width of the rectangle.

$x + 4$

iii) Write the area of the rectangle as a product of two binomials.

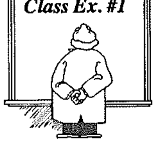
$(x+2)(x+4)$

iv) Write the area of the rectangle in expanded form.

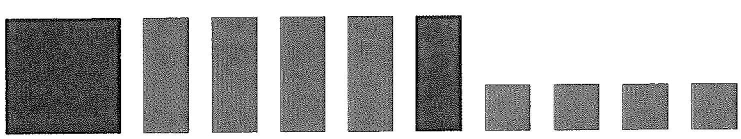
$x^2 + 6x + 8$

- The work above provides a method for factoring the trinomial $x^2 + 6x + 8$ into the product of two binomials $(x + 2)(x + 4)$: i.e. $x^2 + 6x + 8 = (x + 2)(x + 4)$.

Class Ex. #1

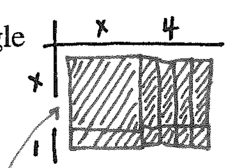


a) Write the polynomial expression which is represented by the algebra tiles.



$x^2 + 5x + 4$

b) Arrange the algebra tiles into a rectangle and write an expression for the length and width of the rectangle.



length = $x + 4$

width = $x + 1$

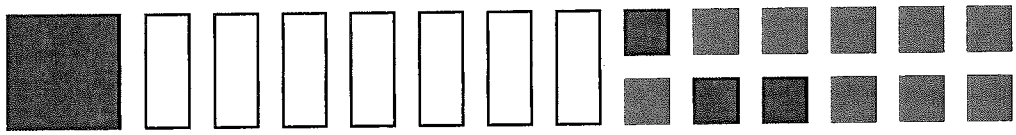
Note: Always place the largest valued tile and then arrange the remaining tiles around.

c) Use the results above to express the polynomial in factored form.

$x^2 + 5x + 4 = (x + 4)(x + 1)$

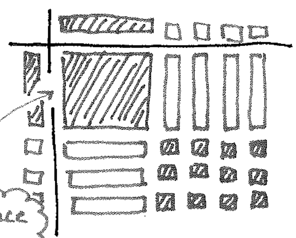
Class Ex. #2

a) Write the polynomial expression which is represented by the algebra tiles.



$x^2 - 7x + 12$

b) Arrange the algebra tiles into a rectangle and express the polynomial in factored form.



$x^2 - 7x + 12 = (x-4)(x-3)$

Always place the largest value tile first and then work outwards!

Complete Assignment Questions #1 - #3

Investigation: Factoring Trinomials by Inspection

- Expand the following binomials as shown.

$(x+2)(x+4) = x^2 + 4x + 2x + 8 = x^2 + 6x + 8$

$(x+3)(x+3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$

$(x+1)(x+7) = x^2 + 7x + x + 7 = x^2 + 8x + 7$

$(x+5)(x+2) = x^2 + 2x + 5x + 10 = x^2 + 7x + 10$

$(x-5)(x-2) = x^2 - 2x - 5x + 10 = x^2 - 7x + 10$

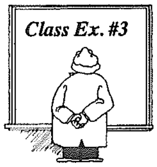
$(x+8)(x-6) = x^2 - 6x + 8x - 48 = x^2 + 2x - 48$

- Consider the expansion $(x+p)(x+q) = x^2 + bx + c$.

  In each of the examples above what is the connection between

  i) the value of b and the values of p and q? $b =$ $p+q$

  ii) the value of c and the values of p and q? $c =$ pq



Use FOIL to show that $(x+p)(x+q)$ can be written in the form $x^2 + (p+q)x + pq$.

F: First
O: Outside
I: Inside
L: Last

$(x)(x) + (x)(q) + (p)(x) + (p)(q)$

$= x^2 + px + qx + pq$ ← arranged parameters in alphabetical order.

$= x^2 + (p+q)x + pq$

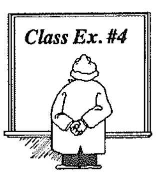
Factoring $x^2 + bx + c$ *by Inspection*

In order to factor $x^2 + bx + c$ by inspection we need to find two integers which have a product equal to c and a sum equal to b. If no two such integers exist, then the polynomial cannot be factored.

In order to factor $x^2 + 8x + 12$ we need to find two numbers which multiply to 12 and add to 8.

In order to factor $x^2 - 13x + 12$ we need to find two numbers which multiply to 12 and add to −13.

The next example practices this skill.



Complete the tables to find two numbers with the given sum and the given product.

Sum	Product	Integers
12	20	2, 10
9	20	4, 5
4	4	2, 2
−9	18	−3, −6

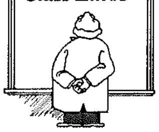
Sum	Product	Integers
−15	14	−1, −14
−1	−6	2, −3
2	−15	−3, 5
−26	48	−2, −24

Notice that:

- if the product is **positive**, then the two integers must be either **both positive** or **both negative.**
- if the product is **negative**, then one integer is **positive** and the other is **negative**.

For the remainder of this lesson, we will only deal with examples where the product is positive. In the next lesson we will include examples where the product is negative.

Class Ex. #5



Factor the following trinomials where possible.

a) $x^2 + 8x + 12$

$= (x+2)(x+6)$

b) $x^2 + 13x + 12$

$= (x+1)(x+12)$

c) $x^2 - 13x + 12$

$= (x-1)(x-12)$

d) $a^2 - 11a + 10$

$= (a-1)(a-10)$

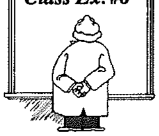
e) $y^2 + 3y + 4$

$=$ not possible

f) $x^2 + 27x + 50$

$= (x+2)(x+25)$

Class Ex. #6



Factor the polynomial expressions by first removing a common factor.

a) $4x^2 - 32x + 48$

$= 4(x^2 - 8x + 12)$

$= 4(x-2)(x-6)$

b) $3x^3 + 21x^2 + 30x$

$= 3x(x^2 + 7x + 10)$

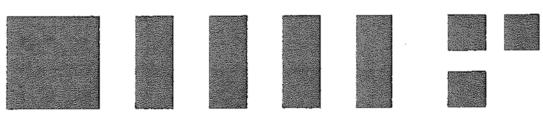
$= 3x(x+2)(x+5)$

In this example there were two steps in the factoring process - a common factor followed by a trinomial. If we are asked to factor a polynomial expression, it is understood this means to continue factoring until no further factoring is possible. This is sometimes written as "factor completely ...". The operation "factor" means "factor completely".

Complete Assignment Questions #4 - #15

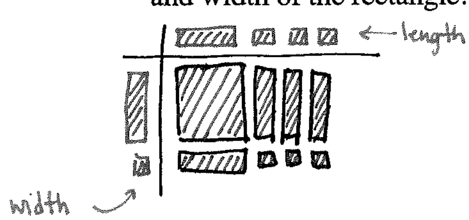
Assignment

1. a) Write the polynomial expression which is represented by the algebra tiles.



$x^2 + 4x + 3$

b) Arrange the algebra tiles into a rectangle and write an expression for the length and width of the rectangle.



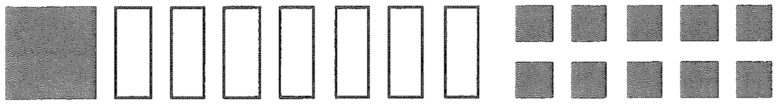
length $= x+3$

width $= x+1$

c) Use the results above to express the polynomial in factored form.

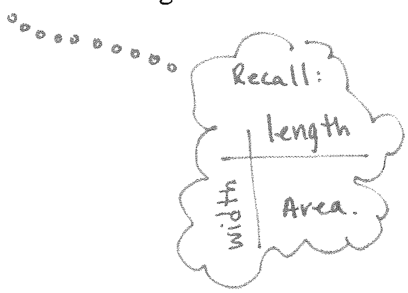
$x^2 + 4x + 3 = (x+3)(x+1)$

2. a) Write a polynomial expression for the group of algebra tiles shown.



$x^2 - 7x - 10$

b) Arrange the algebra tiles into a rectangle.



c) State the length and width of the rectangle and hence express the polynomial in factored form.

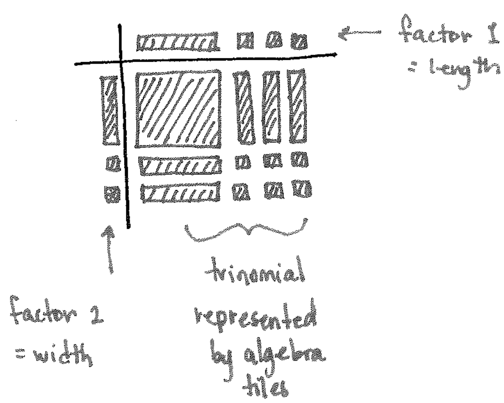
length $= x - 5$

width $= x - 2$

$x^2 - 7x + 10 = (x-5)(x-2)$

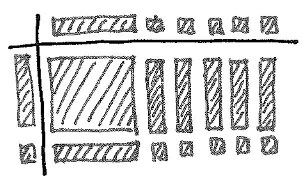
3. Use algebra tiles to factor the following trinomials.

a) $x^2 + 5x + 6$



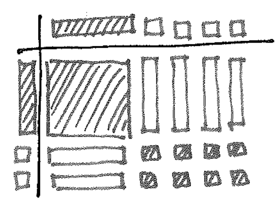
$x^2 + 5x + 6 = \underline{(x+3)(x+2)}$

b) $x^2 + 6x + 5$



$= \underline{(x+5)(x+1)}$

c) $x^2 - 6x + 8$



$= \underline{(x-4)(x-2)}$

4. Complete the tables to find two numbers with the given sum and the given product.

	Sum	Product	Integers
a)	5	6	2, 3
b)	8	7	1, 7
c)	11	30	5, 6
d)	–11	30	–5, –6

	Sum	Product	Integers
e)	11	10	1, 10
f)	–8	15	–3, –5
g)	–15	56	–7, –8
h)	–18	56	–4, –14

Think about it!
factor pairs 30
–1 –30
–2 –15
–3 –10
–5 –6 = –11
~~6 5~~
stop at repeat factors.

Think about it!
factor pairs for 56
–1 –56
–2 –28
–4 –14 = –18
–7 –8
STOP ~~8 7~~

5. Complete the following.

a) $x^2 + 7x + 12 = (x + 3)(x + 4)$
b) $x^2 + 9x + 8 = (x + 1)(x + 8)$
c) $x^2 - 7x + 10 = (x - 2)(x - 5)$
d) $t^2 - 14t + 24 = (t - 2)(t - 12)$
e) $z^2 + 8z + 15 = (z + 5)(z + 3)$
f) $b^2 - 12b + 20 = (b - 2)(b - 10)$

6. Factor the following.

a) $x^2 + 3x + 2$
$= (x+1)(x+2)$

b) $x^2 - 3x + 2$
$= (x-1)(x-2)$

c) $x^2 + 9x + 18$
$= (x+3)(x+6)$

Factor Pairs 18
1 18
2 9
3 6 = 9
~~6 3~~
STOP

d) $x^2 + 8x + 12$
$= (x+2)(x+6)$

e) $x^2 - 10x + 21$
$= (x-3)(x-7)$

f) $x^2 - 11x + 24$
$= (x-3)(x-8)$

7. Factor where possible.

a) $x^2 + 11x + 10$
$= (x+1)(x+10)$

b) $x^2 + 10x + 11$
not possible

c) $n^2 + 12n + 32$
$= (n+4)(n+8)$

d) $y^2 - 11y + 28$
$= (y-4)(y-11)$

e) $y^2 + 17y + 42$
$= (y+3)(y+14)$

f) $f^2 - 10f + 21$
$= (f-3)(f-7)$

g) $p^2 - 16p + 28$
$= (p-2)(p-14)$

h) $x^2 + 24x + 80$
$= (x+4)(x+20)$

i) $c^2 - 32c + 60$
$= (c-2)(c-30)$

j) $a^2 - 12a + 24$
not possible

Factor Pairs 24
–1 –24
–2 –12
–3 –8
–4 –6
~~6 4~~
STOP
problem: no sum of factor pairs add to –12!

k) $d^2 + 18d + 45$
$= (d+3)(d+15)$

l) $p^2 - 29p + 100$
$= (p-4)(p-25)$

m) $m^2 + 22m + 121$
$= (m+11)(m+11)$
or
$= (m+11)^2$

Factor Pair 121
1 121
11 11
Stop repeat

n) $n^2 - 23n + 102$
$= (n-6)(n-17)$

o) $q^2 - 28q + 115$
$= (q-5)(q-23)$

8. a) The expression $x^2 + bx + 12$ can be factored over the integers. Determine all possible values of b.

Goal: To find 2 integers with a product of 12.
$b = 13, 8, 7, -13, -8, -7$

Factor Pairs of 12
$-1 + -12 = -13$
$-2 + -6 = -8$
$-3 + -4 = -7$
negative

Factor Pairs of 12
$1 + 12 = +13$
$2 + 6 = +8$
$3 + 4 = +7$
positive

b) If the expression $x^2 + 6x + c$, where $c > 0$, can be factored over the integers, determine all possible values of c.

Goal: To find 2 positive integers with a sum of 6.
$c = 5, 8, 9$

$\_ + \_ = 6$
$(1)(5) = 5$
$(2)(4) = 8$
$(3)(3) = 9$

9. A volleyball court has an area of $x^2 + 15x + 36$ square metres.

a) Factor $x^2 + 15x + 36$ to find binomials that represent the length and width of the court.

$x^2 + 15x + 36 = (x + 12)(x + 3)$ length $= x + 12$ width $= x + 3$

Factor Pairs 36
1 36
2 18
3 12 $= 15$
4 9
6 6 ← STOP
repeat.

b) If $x = 3$, determine the length and width of the court.

length $= (3) + 12 = 15$m width $= (3) + 3 = 6$ m

10. Factor.

Problem: Each of these polynomials have a leading coefficient. Let us try to remove it as a common factor before factoring the remaing trinomial. If we can it will be a lot easier! Especially if the remaing polynomial can be factored by inspection!

a) $2x^2 + 6x + 4$
$= 2(x^2 + 3x + 2)$
$= 2(x + 1)(x + 2)$

b) $4x^2 - 48x + 128$
$= 4(x^2 - 12x + 32)$
$= 4(x - 4)(x - 8)$

c) $-2a^2 - 30a - 108$
$= -2(a^2 + 15a + 54)$
$= -2(a + 6)(a + 9)$
↑ factor out "–" coefficient

d) $5x^2 - 20x + 15$
$= 5(x^2 - 4x + 3)$
$= 5(x - 1)(x - 3)$

e) $ax^2 - 14ax + 45a$
$= a(x^2 - 14x + 45)$
$= a(x - 5)(x - 9)$

f) $-10a^4 + 100a^3 - 240a^2$
$= -10a^2(a^2 - 10a + 24)$
$= -10a^2(a - 4)(a - 6)$
↑ factor out all variable common factors as well.

11. Consider the following in which each letter represents a whole number.

$x^2 + 5x + 6 = (x + A)(x + B)$ ← B = 3, since b is common →
$= (x+2)(x+3)$ or $(\cancel{x+3})(x+2)$

$x^2 + 10x + 21 = (x + B)(x + G)$
$= (x+3)(x+7)$ or $(\cancel{x+7})(x+3)$

$x^2 - 9x + 20 = (x - T)(x - L)$
$= (x-4)(x-5)$ or $(\cancel{x-5})(x-4)$
because T must = 4

$2x^2 - 16x + 32 = 2(x - T)^2$
$= 2(x^2 - 8x + 16) = 2(x-4)^2$
Must be T

$x^3 + 10x^2 + 9x = x(x + S)(x + E)$ ← S = 1, common →
$= x(x^2 + 10x + 9) = x(x+9)(\cancel{x+1})$
or
$x(x+1)(x+9)$ → E

$6x^2 - 54x + 48 = 6(x - I)(x - S)$
$= 6(x^2 - 9x + 8) = 6(\cancel{x-1})(\cancel{x-8})$
or $6(x-8)(x-1)$

1 = S
2 = A
3 = B
4 = T
5 = L
7 = G
8 = I
9 = E

Determine the value of each letter and hence name the famous person represented by the following code.

(3)	(8)	(5)	(5)	(7)	(2)	(4)	(9)	(1)
B	I	L	L	G	A	T	E	S

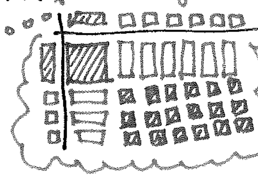
Multiple Choice

12. Which of the following is **not** a factor of $3m^2 - 27m + 54$?

A. $m - 3$
B. $m - 6$
C. $m - 9$
D. 3

STEP 1: Leading coeficient is not equal to 1, so attempt to factor out a 3.
$= 3(m^2 - 9m + 18)$
STEP 2: Factor using inspection
$= 3(m-6)(m-3)$

Note: Could us an area diagram



13. For which of the following trinomials is $a + 5$ **not** a factor?

A. $a^2 + 6a + 5$ $= (a+5)(a+1)$
B. $a^2 + 11a + 30$ $= (a+5)(a+6)$
C. $a^2 + 10a + 50$ $=$ (not possible to factor)
D. $a^2 + 10a + 25$ $= (a+5)^2$

Factor Pairs of 50 →
$1 + 50 = 51$
$2 + 25 = 27$
$5 + 10 = 15$
~~10~~ ~~5~~ ← STOP

14. The expression $t^2 + kt + 12$ **cannot** be factored if k has the value

A. -13 — $t + (-13)t + 12$ $= t - 13t + 12$ $= (t-1)(t-12)$
B. -8 — $t + (-8)t + 12$ $= t - 8t + 12$ $= (t-2)(t-6)$
C. 7 — $t + (7)t + 12$ $= t + 7t + 12$ $= (t+3)(t+4)$
D. 11 — $t + (11)t + 12$ $= t + 11t + 12$ PROBLEM

11
1, 12 = 13
2, 6 = 8
3, 4 = 7
~~4, 3~~ = 7

Numerical Response

15. The largest value of b (positive?) for which $x^2 + bx + 32$ can be factored over the integers is ____.

(Record your answer in the numerical response box from left to right)

3	3		

$(x+1)(x+32) = x^2 + 33x + 32$

Factor Pairs. 32
$+1 + +32 = 33$
$+2 + +16 = 18$
$+4 + +8 = 12$
~~8 4~~

Answer Key

1. a) $x^2 + 4x + 3$ **b)** $x + 3,\ x + 1,$ **c)** $x^2 + 4x + 3 = (x + 3)(x + 1)$
2. a) $x^2 - 7x + 10$ **c)** $x - 2, x - 5,$ $x^2 - 7x + 10 = (x - 2)(x - 5)$
3. a) $(x + 2)(x + 3)$ **b)** $(x + 1)(x + 5)$ **c)** $(x - 4)(x - 2)$
4. a) 2, 3 **b)** 1, 7 **c)** 5, 6 **d)** –5, –6
e) 1, 10 **f)** –3, –5 **g)** –7, –8 **h)** –4, –14
5. a) $(x + 3)(x + 4)$ **b)** $(x + 1)(x + 8)$ **c)** $(x - 2)(x - 5)$
d) $(t - 2)(t - 12)$ **e)** $(z + 5)(z + 3)$ **f)** $(b - 2)(b - 10)$

6. a) $(x + 1)(x + 2)$ **b)** $(x - 1)(x - 2)$ **c)** $(x + 3)(x + 6)$
d) $(x + 2)(x + 6)$ **e)** $(x - 3)(x - 7)$ **f)** $(x - 3)(x - 8)$

7. a) $(x + 1)(x + 10)$ **b)** not possible **c)** $(n + 4)(n + 8)$
d) $(y - 4)(y - 7)$ **e)** $(y + 3)(y + 14)$ **f)** $(f - 3)(f - 7)$
g) $(p - 2)(p - 14)$ **h)** $(x + 4)(x + 20)$ **i)** $(c - 2)(c - 30)$
j) not possible **k)** $(d + 3)(d + 15)$ **l)** $(p - 4)(p - 25)$
m) $(m + 11)(m + 11)$ OR $(m + 11)^2$ **n)** $(n - 6)(n - 17)$ **o)** $(q - 5)(q - 23)$
8. a) 7, 8, 13,–7, –8, –13 **b)** 5, 8, 9 **9. a)** $(x + 12)(x + 3)$ **b)** 15m, 6m

10. a) $2(x + 1)(x + 2)$ **b)** $4(x - 4)(x - 8)$ **c)** $-2(a + 6)(a + 9)$
d) $5(x - 1)(x - 3)$ **e)** $a(x - 5)(x - 9)$ **f)** $-10a^2(a - 4)(a - 6)$
11. BILL GATES **12.** C **13.** C **14.** D **15.**

3	3		

Factoring Polynomial Expressions Lesson #3:
Factoring Trinomials of the Form $x^2 + bx + c$ - Part Two

Review of Factoring By Inspection

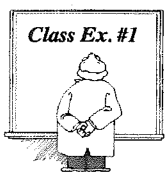
In order to factor $x^2 + bx + c$ by inspection, we need to find two integers which have a <u>product equal to c</u> and a <u>sum equal to b</u>. If no two such integers exist, then the polynomial cannot be factored.

In order to factor $x^2 + 6x + 9$, we need to find two numbers whose product is ___ and whose sum is _____ .

In order to factor $x^2 + x - 12$, we need to find two numbers whose product is ___ and whose sum is _____ .

Recall the following points from the previous lesson.

- If the product is **positive**, then the two integers must be either **both positive** or **both negative.**
- If the product is **negative**, then one integer is **positive** and the other is **negative**.



Class Ex. #1

Factor the following trinomials by inspection.

a) $x^2 - x - 12$ $[\pm 1, \pm 12;\ \pm 2, \pm 6;\ \pm 3, \pm 4]$

$(3)(-4) = -12$

$(3) + (-4) = -1$

$\boxed{(x+3)(x-4)}$

b) $x^2 + 3x - 18$ $[\pm 1, \pm 18;\ \pm 2, \pm 9;\ \pm 3, \pm 6]$

$(6)(-3) = -18$

$(6) + (-3) = 3$

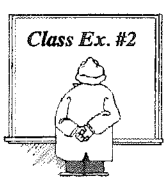
$\boxed{(x+6)(x-3)}$

c) $a^2 - 7a - 8$ $[\pm 1, \pm 8;\ \pm 2, \pm 4]$

$(-8)(1) = -8$

$(-8) + (1) = -7$

$\boxed{(x-8)(x+1)}$



Class Ex. #2

Factor where possible.

a) $-a^2 - 6a + 27$ $[\pm 1, \pm 27;\ \pm 2, \pm 3]$

$(-1)(a^2 + 6a - 27)$

$(-3)(+9) = -27$

$(-3) + (+9) = 6$

$\boxed{-1(a-3)(a+9)}$

b) $2t^2 - 14t + 20$ $[\pm 1, \pm 10;\ \pm 2, \pm 5]$

$2(t^2 - 7t + 10)$

$(-2)(-5) = 10$

$(-2) + (-5) = -7$

$\boxed{2(t-2)(t-5)}$

c) $x^2 - 3x - 6$ $[\pm 1, \pm 6]$

$(?)(?) = -6$

$(?) + (?) = -3$

There are NO two identical numbers which both multiply to -6 and add up to -3, $\therefore$ factoring is Not possible in $x \in I$

d) $4x^4 - 16x^3 - 20x^2$

$4x^2(x^2 - 4x - 5)$ $[\pm 1, \pm 5]$

$(1)(-5) = -5$

$(1) + (-5) = -4$

$\boxed{4x^2(x+1)(x-5)}$

Complete Assignment Questions #1 - #5

Factoring Trinomials of the form $x^2 + bxy + cy^2$

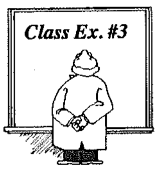
Complete the following statements:

i) $(x + 2)(x + 4)$ can be expanded to $x^2 + 6x + 8$,
so $x^2 + 6x + 8$ can be factored to ($x+2$)($x+4$).

ii) $(x + 2y)(x + 4y)$ can be expanded to $x^2 + 6xy + 8y$,
so $x^2 + 6xy + 8y$ can be factored to ($x+2y$)($x+4y$).

Recall:

	x	$2y$
x	x^2	$2xy$
$4y$	$4xy$	$8y^2$



Class Ex. #3

Factor.

a) $x^2 + 13xy + 30y^2$
$= (x+3y)(x+10y)$

b) $x^2 + 71xy - 72y^2$
$= (x-y)(x-72y)$

c) $3a^2 - 15ab - 252b^2$
$= 3(a^2 - 5ab - 84b^2)$
$= 3(a+7b)(a-12b)$

Complete Assignment Questions #6 - #11

Assignment

Factor Pair

-20
$-1 \quad +20$
$-2 \quad +10 = 8$
$-4 \quad +5$
~~5 -4~~
↑ Stop

$(-)(+)$ ↓ middle term

1. Complete the table to find two numbers with the given sum and the given product.

	Sum	Product	Integers
a)	8	−20	−2, 10
b)	−8	−20	2, −10
c)	−1	−20	4, −5

	Sum	Product	Integers
d)	3	−70	−7, 10
e)	−11	28	−4, −7
f)	0	−16	−4, 4

2. Factor the following trinomials.

a) $x^2 - 2x - 15$
$= (x-5)(x+3)$

b) $x^2 - 2x - 24$
$= (x-6)(x+4)$

c) $x^2 + 2x - 24$
$= (x+6)(x-4)$

d) $x^2 + 2x - 3$
$= (x+3)(x-1)$

e) $x^2 + x - 30$
$= (x+6)(x-5)$

f) $x^2 - 3x - 10$
$= (x-5)(x+2)$

$(-)(+)$ ↓ larger absolute value of integer given middle term.
-24
$-1 \quad +24$
$-2 \quad +12$
$-3 \quad +8$
$-4 \quad +6$
~~6 -4~~
↑ Stop

$(-)(+)$ ↓ larger |integer| from factor pair given middle term.
-15
$+1 \quad -15 = -14$
$+3 \quad -5 = -2$
~~5 -3~~

Factor Pair
16
+1 +16
+2 +8 = 10
+4 +4 ← STOP
repeat

3. Factor where possible.

a) $x^2 + 10x + 16$
$= (x+2)(x+8)$

b) $x^2 - 11x + 18$
$= (x-2)(x-9)$

c) $x^2 - 2x - 8$
$= (x+2)(x-4)$

d) $x^2 + 3x - 18$
$= (x+6)(x-3)$

e) $x^2 - 4x + 12$
= not possible

f) $x^2 - 4x - 12$
$= (x-6)(x+2)$

g) $x^2 - 10x + 25$
$= (x-5)(x-5)$
or
$= (x-5)^2$

h) $x^2 + x - 20$
$= (x+5)(x-4)$

Recall: $(-)(+)$ larger |integer| given sign of middle term.

Factor Pairs
12
$-1 \quad -12 = -13$
$-2 \quad -6 = -8$
$-3 \quad -4 = -7$
~~4 3~~
STOP

i) $m^2 + 21m + 38$
$= (m+2)(m+19)$

j) $a^2 - 17a + 42$
$= (a-14)(a-3)$

k) $p^2 - 10p - 9$
= not possible

l) $p^2 - 9p - 10$
$= (p-10)(p+1)$

4. Factor.

a) $-x^2 - 7x - 12$
$= -1(x^2 + 7x + 12)$
$= -1(x+3)(x+4)$

b) $4x^2 - 28x - 32$
$= 4(x^2 - 7x - 8)$
$= 4(x-8)(x+1)$

c) $5x^2 - 20x + 15$
$= 5(x^2 - 4x + 3)$
$= 5(x-3)(x-1)$

Factor Pair
-10
$+1 \quad -10 = -9$
$+2 \quad -5$
~~5 2~~
STOP

d) $-2a^2 + 2a + 220$
$= -2(a^2 - a - 110)$
$= -2(a-11)(a+10)$

e) $b^2x^2 - 4b^2x - 45b^2$
$= b^2(x^2 - 4x - 45)$
$= b^2(x-9)(x+5)$

f) $2x^3 + 2x^2 - 40x$
$= 2x(x^2 + x - 20)$
$= 2x(x+5)(x-4)$

Note: When factoring any polynomial always attempt to factor out a greatest common factor before moving on to inspection or any other type of more complex factoring. This includes factoring out a negative from the leading coefficient.

5. Consider the following in which the each letter represents a whole number.

$x^2 + 4x - 5 = (x + A)(x - O)$
$= (x+5)(x-1)$

$x^2 - 3x - 54 = (x - E)(x + I)$
$= (x-9)(x+6)$

$x^3 + 2x^2 - 8x = x(x - Y)(x + P)$
$= x(x^2 + 2x - 8) = x(x-2)(x+4)$

$3x^2 - 48x + 192 = T(x - R)^2$
$= 3(x^2 - 16x + 64) = 3(x-8)^2$

$-5x^2 + 20x + 105 = -5(x + T)(x - H)$
$= -5(x^2 - 4x - 21) = -5(x+3)(x-7)$

1 = O
2 = Y
3 = T
4 = P
5 = A
7 = H
8 = R
9 = E

Determine the value of each letter and hence name the fictional character represented by the following code.

(7)	(5)	(8)	(8)	(2)	(4)	(1)	(3)	(3)	(9)	(8)
H	A	R	R	Y	P	O	T	T	E	R

6. Factor.

a) $x^2 + 18xy + 45y^2$
$= (x+15y)(x+3y)$

b) $x^2 + 10xy - 24y^2$
$= (x-2y)(x+12y)$

c) $a^2 - 12ab + 36b^2$
$= (a-6b)(a-6b)$ or $(a-6b)^2$

d) $p^2 - 12pq + 11q^2$
$= (p-q)(p-11q)$

e) $x^2 + xy - 72y^2$
$= (x-8y)(x+9y)$

f) $x^2 - 54xy - 112y^2$
$= (x+2y)(x-56y)$

7. Factor completely.

a) $4x^2 - 80xy + 144y^2$
$= 4(x^2 - 20xy + 36y^2)$
$= 4(x-18y)(x-2y)$

b) $3b^2 - 15bv - 72v^2$
$= 3(b^2 - 5bv - 24v^2)$
$= 3(b-8v)(b+3v)$

c) $2c^2 + 66cd - 140d^2$
$= 2(c^2 + 33cd - 70d^2)$
$= 2(c+35d)(c-2d)$

Multiple Choice

8. When factored, the trinomials $x^2 - 10x + 21$ and $x^2 - 4x - 21$ have one binomial factor in common. This factor is

A. $x - 7$ **B.** $x + 7$
C. $x - 3$ **D.** $x + 3$

$x^2 - 10x + 21 = (x-3)(x-7)$
$x^2 - 4x - 21 = (x+3)(x-7)$

9. One factor of $-m^3 - m^2 + 6m$ is

A. $m - 2$ **B.** $m + 2$
C. $m - 3$ **D.** $m - 6$

Notice there is a m variable in all 3 terms.
Also notice that the leading coefficient is "−".
$= -m(m^2 + m - 6)$
$= -m(m+3)(m-2)$

10. One factor of $3x^2 - 6xy - 9y^2$ is

A. $3x$ **B.** $x + 2y$
C. $x + 3y$ **D.** $x + y$

$= 3(x^2 - 2xy - 3y^2)$
$= 3(x-3y)(x+y)$

11. The expression $x^2 - 4x + c$ **cannot** be factored if c has the value

A. −5 $x^2 - 4x - 5 = (x-5)(x+1)$
B. 0 $x^2 - 4x = x(x-4)$
C. 4 $x^2 - 4x + 4 = (x-2)^2$
D. 5 $x^2 - 4x + 5 =$ not possible.

Factor Pairs.
5
$-1 \quad -5 = -6$
~~5 1~~
STOP

Answer Key

1. a) −2, 10 b) −10, 2 c) −5, 4 d) −7, 10 e) −4, −7 f) −4, 4
2. a) $(x-5)(x+3)$ b) $(x-6)(x+4)$ c) $(x+6)(x-4)$ d) $(x+3)(x-1)$
e) $(x+6)(x-5)$ f) $(x-5)(x+2)$
3. a) $(x+8)(x+2)$ b) $(x-9)(x-2)$ c) $(x+2)(x-4)$ d) $(x+6)(x-3)$
e) not possible f) $(x-6)(x+2)$ g) $(x-5)^2$ h) $(x+5)(x-4)$
i) $(m+2)(m+19)$ j) $(a-14)(a-3)$ k) not possible l) $(p-10)(p+1)$
4. a) $-(x+3)(x+4)$ b) $4(x-8)(x+1)$ c) $5(x-3)(x-1)$ d) $-2(a-11)(a+10)$
e) $b^2(x-9)(x+5)$ f) $2x(x+5)(x-4)$ **5.** HARRY POTTER
6. a) $(x+15y)(x+3y)$ b) $(x-2y)(x+12y)$ c) $(a-6b)^2$ d) $(p-q)(p-11q)$
e) $(x-8y)(x+9y)$ f) $(x+2y)(x-56y)$
7. a) $4(x-18y)(x-2y)$ b) $3(b-8v)(b+3v)$ c) $2(c+35d)(c-2d)$
8. A **9.** A **10.** D **11.** D

Factoring Polynomial Expressions Lesson #4:
Difference of Squares

Investigation

a) Complete the following using the trinomial factoring method from the previous lessons.

	Sum	Product	Integers	Polynomial	Factored Form
i)	–6	–16	$-8, 2$	$x^2 - 6x - 16$	$(x-8)(x+2)$
ii)	–15	–16	$-16, 1$	$x^2 - 15x - 16$	$(x-16)(x+1)$
iii)	0	–16	$-4, 4$	$x^2 + 0x - 16 = x^2 - 16$	$(x-4)(x+4)$
iv)	0	–64	$-8, 8$	$x^2 + 0x - 64 = x^2 - 64$	$(x-8)(x+8)$
v)	0	–25	$-5, 5$	$x^2 + 0x - 25 = x^2 - 25$	$(x-5)(x+5)$

$-5x + 5x = 0x$

b) The third row in a) shows that the factored form of $x^2 - 16$ is $(x-4)(x+4)$. Use the pattern from the last three rows to factor the following.

i) $x^2 - 9 =$

$(x-3)(x+3)$

ii) $x^2 - 49 =$

$(x-7)(x+7)$

iii) $x^2 - 36 =$

$= (x-6)(x+6)$

iv) $x^2 - 1 =$

$= (x-1)(x+1)$

v) $a^2 - 100 =$

$(a-10)(a+10)$

c) Extend the procedure from above to factor $m^2 - n^2$.
Verify your answer by expanding the factored form.

$\underline{(m-n)(m+n)}$

Check Work.

	m	$-n$
m	m^2	$-mn$
n	mn	$-n^2$

$m^2 + 0mn - n^2$
$= \underline{m^2 - n^2}$

d) Consider the expansion $(x-y)(x+y) = x^2 + bx + c$.

i) Explain why the value of b is zero.

$(x-y)(x+y) = x^2 + xy - xy - y^2$

The two middle terms in the expansion are opposites of each other. There sum is zero so $b = 0$.

ii) Express c in terms of y.

$c = -y^2$

Difference of Squares

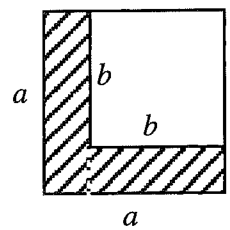
The examples on the previous page are trinomials of the form $x^2 + bx + c$, where $b = 0$ and c is the negative of a square number.

This results in a **difference of squares** such as $x^2 - 25$, $x^2 - 100$, etc.

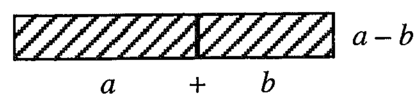
To factor a difference of squares we can use the identity:

$$\boxed{a^2 - b^2 = (a - b)(a + b)}$$

The identity $a^2 - b^2 = (a - b)(a + b)$ can be illustrated in the following diagram.

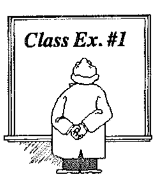


Shaded area $= a^2 - b^2$



Shaded area $= (a - b)(a + b)$

The shaded area on the left is cut along the dotted line and rearranged to form the diagram on the right.
The shaded area on the left is represented by $a^2 - b^2$ *and the shaded area on the right is represented by* $(a - b)(a + b)$.



Factor the following polynomials using the difference of squares method.

a) $a^2 - 4$

$= (a-2)(a+2)$

b) $t^2 - 144$

$= (t-12)(t+12)$

c) $x^2 - y^2$

$= (x-y)(x+y)$

d) $p^2 - 7^2$

$= (p-7)(p+7)$

Note that it is not possible to factor a sum of squares like $x^2 + 4$, i.e. $x^2 + 0x + 4$.
It is not possible to find two integers whose product is positive and whose sum is zero.

In the identity $a^2 - b^2 = (a - b)(a + b)$ we can replace a and/or b by numbers, variables, monomials and even polynomials.

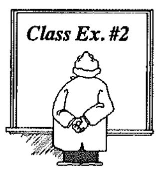
For example, $4x^2 - 25$ can be written as $(2x)^2 - (5)^2$ and can be factored using the above identity with $a = 2x$ and $b = 5$.

$$4x^2 - 25 = (\ 2x - 5\ )(\ 2x + 5\ )$$

$9m^2 - 4n^2$ can be written as $(3m)^2 - (2n)^2$, and can be factored using the above identity with $a = 3m$ and $b = 2n$.

$$9m^2 - 4n^2 = (\ 3m - 2n)(\ 3m + 2n\ )$$

The factoring above can be verified by expanding the product of the factors.



Factor, if possible, using the difference of squares method.

a) $16t^2 - 49$

$= (4t - 7)(4t + 7)$

b) $81a^2 - 1$

$= (9a - 1)(9a + 1)$

c) $100 - y^2$

$= (10 - y)(10 + y)$

d) $36p^2 - 25q^2$

$= (6p - 5q)(6p + 5q)$

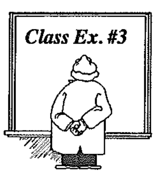
e) $4x^2 + 25$

$=$ not possible

f) $64 - 9a^2b^2$

$= (8 - 3ab)(8 + 3ab)$

PROBLEM: Difference of squares implies subtraction.



The floor of an international doubles squash court is rectangular with an area of $25a^2 - b^2$ square feet.

a) Write expressions for the length and width of the floor.

length $= 5a + b$ ft width $= 5a - b$ ft

b) The perimeter of the floor is 140 feet. Determine the length and width of the floor if the length is 1.8 times the width.

Step 1: Relate length and width to perimeter to hopefully solve for at least one variable.

Perimeter $= 2$ length $+ 2$ width $= 140$

Perimeter $= 2(5a + b) + 2(5a - b) = 140$

$10a + 2b + 10a - 2b = 140$

$20a = 140$

$\boxed{a = 7}$

S2: Relate length to width to hopefully solve for b.

$5a + b = 1.8(5a - b)$

$5a + b = 9a - 1.8b$

$2.8b = 4a$

Let $\boxed{a = 7}$

$2.8b = 4(7) = 28$

$b = \dfrac{28}{2.8} = \boxed{10}$

Thus,

length $= 5(7) + 10$

$= 45$ ft

width $= 5(7) - 10$

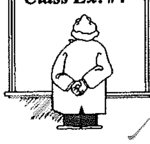
$= 25$ ft

Difference of Squares involving a Common Factor

The first step in factoring any polynomial expression should be to determine if we can remove a common factor.

Factor the following polynomials by first removing the greatest common factor.

Class Ex. #4



a) $2a^2 - 50$
$= 2(a^2 - 25)$
$= 2(a-5)(a+5)$

b) $3x^2 - 12y^2$
$= 3(x^2 - 4y^2)$
$= 3(x-2y)(x+2y)$

c) $144p^2q^2 - 4$
$= 4(36p^2q^2 - 1)$
$= 4(6pq-1)(6pq+1)$

d) $3x^3 - 27x$
$= 3x(x^2 - 9)$
$= 3x(x-3)(x+3)$

Notice the leading coefficiant has a value other than positive 1.

Complete Assignment Questions #1 - #14

Assignment

1. Complete the following by determining the missing factor.

a) $x^2 - 36 = (x-6)(\ x+6\ )$ **b)** $c^2 - 121 = (c+11)(\ c-11\ )$ **c)** $j^2 - k^2 = (j-k)(\ j+k\ )$

2. Factor the following polynomials using a difference of squares.

a) $x^2 - 49$ $= (x-7)(x+7)$

b) $x^2 - 1$ $= (x-1)(x+1)$

c) $x^2 - 15^2$ $= (x-15)(x+15)$

d) $x^2 - 400$ $= (x-20)(x+20)$

3. Explain how factoring a difference of squares in one variable can be regarded as a special case of factoring trinomials by inspection.

A difference of squares can be regarded as a trinomial of the form x^2+bx+c in which $b=0$ and c is negative. We need to find two numbers which multiply to c and add to zero.

4. Factor where possible.

a) $m^2 - n^2$ $= (m-n)(m+n)$

b) $c^2 - 7^2$ $= (c-7)(c+7)$

c) $1 - k^2$ $= (1-k)(1+k)$

d) $g^2 - 64h^2$ $= (g-8h)(g+8h)$

e) $25x^2 - 144$ $= (5x-12)(5x+12)$

f) $16a^2 - 9b^2$ $= (4a-3b)(4a+3b)$

g) $4x^2 + z^2$ = not possible

h) $121a^2 - 36b^2$ $= (11a-6b)(11a+6b)$

i) $49 - 4h$ = not possible

$\sqrt{49} = 7$ ✓✓

$\sqrt{4h} = ?$

Problem: "h" is not squared.

j) $100 - 81b^2$ $= (10-9b)(10+9b)$

k) $1 - 25z^2$ $= (1-5z)(1+5z)$

l) $225a^2 - b^2$ $= (15a-b)(15a+b)$

m) $169z^2 - 4q^2$ $= (13z-2q)(13z+2q)$

n) $256 - y^2$ $= (16-y)(16+y)$

o) $t^2 + 36z^2$ = not possible

p) $49a^2 - 400$ $= (7a-20)(7a+20)$

5. The floor of a classroom is rectangular with an area of $81m^2 - 4n^2$ square metres.

a) Write expressions in m and n for the length and width of the floor.

length $= 9m + 2n$ metres   width $= 9m - 2n$ metres

b) If the perimeter of the floor is 72 metres, form an equation in m and n and solve for m.

$$\text{Perimeter} = 2(9m+2n) + 2(9m-2n) = 72$$
$$18m + 4n + 18m - 4n = 72$$
$$\frac{36m}{36} = \frac{72}{36}$$
$$m = \frac{72}{36}$$
$$m = 2$$

c) Determine the length and width of the floor if the length is 25% greater the width.

$$9m + 2n = 1.25(9m - 2n)$$
Let $m = 2$
$$9(2) + 2n = 1.25(9(2) - 2n)$$
$$18 + 2n = 1.25(18 - 2n)$$
$$18 + 2n = 22.5 - 2.5n$$
$$4.5n = 4.5$$
$$\boxed{n = 1}$$

length $= 9(2) + 2(1) = 20$ m

width $= 9(2) - 2(1) = 16$ m

Remember: Leading coefficient is other than ±1. Let us first check if there are any greatest common factors to factor out first before checking for difference of squares.

6. Factor.

a) $8x^2 - 32$
$= 8(x^2 - 4)$
$= 8(x-2)(x+2)$

b) $4a^2 - 100y^2$
$= 4(a^2 - 25y^2)$
$= 4(a-5y)(a+5y)$

c) $3t^2 + 27s^2$
$= 3(t^2 + 9s^2)$

STOP: Though the first and last terms are squares the binomial is a sum.

d) $7x^2 - 7y^2$
$= 7(x^2 - y^2)$
$= 7(x-y)(x+y)$

e) $9a^2b^2 - 36$
$= 9(a^2b^2 - 4)$
$= 9(ab-2)(ab+2)$

f) $8 - 50p^2q^2$
$= 2(4 - 25p^2q^2)$
$= 2(2-5pq)(2+5pq)$

g) $xy^2 - x^3$
$= x(y^2 - x^2)$
$= x(y-x)(y+x)$

h) $20a^2b^2 - 5a^4b^4$
$= 5a^2b^2(4 - a^2b^2)$
$= 5a^2b^2(2-ab)(2+ab)$

7. Factor.

a) $a^2b^2 - 9$
$= (ab-3)(ab+3)$

b) $c^2 - d^2e^2$
$= (c-de)(c+de)$

c) $100x^2 - y^2z^2$
$= (10x-yz)(10x+yz)$

d) $p^2q^2 - r^2s^2$
$= (pq-rs)(pq+rs)$

e) $25x^2y^2 - 1$
$= (5xy-1)(5xy+1)$

f) $c^2d^2 - 4f^2$
$= (cd-2f)(cd+2f)$

g) $4x^2a^2 - 49z^2t^2$
$= (2xa - 7zt)(2xa + 7zt)$

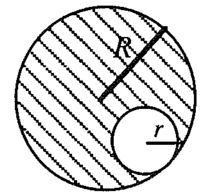
h) $16a^2c^2 - 225b^2d^2$
$= (4ac - 15bd)(4ac + 15bd)$

Think about it! This question is just as simple as $x^2 - 4 = (x-2)(x+2)$. Let us take a look.

$\sqrt{16a^2c^2} = \boxed{4ac}$ $\sqrt{x^2} = x$

$\sqrt{225b^2d^2} = \boxed{15bd}$ $\sqrt{4} = 2$

8. The diagram shows a circle of radius R with a circle of radius r removed.



a) Write an expression for the shaded area.

area $= \pi R^2 - \pi r^2$ ← Is a difference of squares, but first must remove common factor π.

b) Write the expression in a) in factored form.

area $= \pi(R^2 - r^2) = \pi(R-r)(R+r)$

c) Determine the shaded area (as a multiple of π) if $R = 8.5$ and $r = 1.5$. Do not use a calculator.

area $= \pi(8.5-1.5)(8.5+1.5)$
$= \pi(7)(10)$
$= 70\pi$

9. The expression $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$ occurs in physics.

a) Write the expression in factored form.

$= \frac{1}{2}m(v^2-u^2) = \frac{1}{2}m(v-u)(v+u)$

Is a difference of squares, but first must factor out the common factor of $\frac{1}{2}m$.

b) Determine the value of the expression when $m = 10$, $v = 75$, and $u = 25$. Do not use a calculator.

$\frac{1}{2}(10)(75-25)(75+25)$

$= \frac{1}{2}(10)(50)(100)$

$= (5)(50)(100) = (250)(100) = 25000$

10. Consider the following in which each letter represents a whole number.

$64x^2 - y^2 = (Hx - y)(Hx + y)$
$= (8x-y)(8x+y)$

$16x^2 - 4 = C(Ix + 1)(Ix - 1)$
$= 4(x^2-1)$
$= 4(x-1)(x+1) = 4(x+1)(x-1)$

$7x^2 - 252y^2 = P(x - Ey)(x + Ey)$
$= 7(x^2-36y^2)$
$= 7(x-6y)(x+6y)$

$Lx^2 - Ny^2 = (3x - 5y)(Sx + Ay)$
$9x^2 - 25y^2 = (3x-5y)(3x+5y)$

1 = I
3 = S
4 = C
5 = A
6 = E
7 = P
8 = H
9 = L
25 = N

Determine the value of each letter and hence name the country represented by the following code.

(4)	(8)	(2)	(9)	(6)
C	H	I	L	E

11. Susan was showing Rose how the difference of squares method can be used to multiply certain numbers without using a calculator. She showed Rose the following:

$$38 \times 42$$
$$= (40-2)(40+2) \quad = (40^2 - 2^2) \quad = (1600 - 4) \quad = 1596$$

Think about it! The value of 2 is not random, but the mid-point between 38 and 42.

38 (40) 42

a) Use the above process to evaluate:

i) 27×33

27 (30) 33

$= (30-3)(30+3)$
$= 30^2 - 3^2$
$= 900 - 9$
$= 891$

ii) 61×59

59 (60) 61

$= (60+1)(60-1)$
$= (60^2 - 1^2)$
$= 3600 - 1$
$= 3599$

b) Explain why this process is more difficult to determine the product 66×72.

66 (69) 72 problem!

66×72 expressed as a difference of squares $(69^2 - 3^2)$ cannot easily be evaluated without a calculator or long multiplication.

c) Make up your own multiplication question which can be answered using this process.

Evaluate 42×38 without using long multiplication or a calculator.

Multiple Choice

12. One factor of $16 - 4m^2$ is

$= 4(4 - m^2)$
$= 4(2-m)(2+m)$

A. $4 - m$
B. $8 - 2m$
C. $4 + m$
D. $2 + m$

13. Given that $x^2 - y^2 = 45$ and $x + y = 9$, the value of x is

A. 2
B. 5
C. 7
D. impossible to determine

$x^2 - y^2 = (x-y)(x+y)$
$45 = (x-y)(9)$
$5 = (x-y)$

Thus, $x - y = 5$ and $x + y = 9$
so $x = 7$ and $y = 2$
just by inspection.

Numerical Response

14. $3x + 2y$ is a factor of the binomial $a^2x^2 - b^2y^2$. The value of $a^2 + b^2$ is ______ .
(Record your answer in the numerical response box from left to right)

1	3		

$(3x - 2y)(3x + 2y) = 9x^2 - 4y^2$

$a^2 = 9 \quad b^2 = 4$

$a^2 + b^2 = 9 + 4 = 13$

Answer Key

1. a) $(x+6)$ b) $(c-11)$ c) $(j+k)$

2. a) $(x-7)(x+7)$ b) $(x-1)(x+1)$ c) $(x-15)(x+15)$ d) $(x-20)(x+20)$

3. A difference of squares can be regarded as a trinomial of the form x^2+bx+c in which $b=0$ and c is negative. We need to find two numbers which multiply to c and add to zero.

4. a) $(m-n)(m+n)$ b) $(c-7)(c+7)$ c) $(1-k)(1+k)$
d) $(g-8h)(g+8h)$ e) $(5x-12)(5x+12)$ f) $(4a-3b)(4a+3b)$
g) not factorable h) $(11a-6b)(11a+6b)$ i) not factorable using whole number exponent.
j) $(10-9b)(10+9b)$ k) $(1+5z)(1-5z)$ l) $(15a+b)(15a-b)$
m) $(13z-2q)(13z+2q)$ n) $(16-y)(16+y)$ o) not factorable p) $(7a+20)(7a-20)$

5. a) $(9m+2n)$ metres, $(9m-2n)$ metres b) $2(9m+2n)+2(9m-2n)=72$, $m=2$
c) Length = 20 metres, Width = 16 metres.

6. a) $8(x-2)(x+2)$ b) $4(a-5y)(a+5y)$ c) $3(t^2+9s^2)$ d) $7(x-y)(x+y)$
e) $9(ab-2)(ab+2)$ f) $2(2-5pq)(2+5pq)$ g) $x(y-x)(y+x)$ h) $5a^2b^2(2-ab)(2+ab)$

7. a) $(ab-3)(ab+3)$ b) $(c-de)(c+de)$ c) $(10x-yz)(10x+yz)$
d) $(pq-rs)(pq+rs)$ e) $(5xy-1)(5xy+1)$ f) $(cd-2f)(cd+2f)$
g) $(2xa-7zt)(2xa+7zt)$ h) $(4ac-15bd)(4ac+15bd)$

8. a) $A=\pi R^2-\pi r^2$ b) $\pi(R-r)(R+r)$ c) 70π

~~9. a) $\frac{1}{2}m(v-u)(v+u)$ b) 25 000~~ 10. CHILE

11. a) i) 891 ii) 3599
b) 66×72 expressed as a difference of squares (69^2 - 3^2) cannot easily be evaluated without a calculator or long multiplication.

12. D 13. C 14.

1	3		

Factoring Polynomial Expressions Lesson #5: Factoring Review

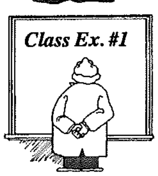
Guidelines for Factoring a Polynomial Expression

If we are asked to factor a polynomial expression, the following guidelines should help us to determine the best method.

1. Look for a common factor. If there is one, take out the common factor and look for further factoring.
2. If there is a binomial expression, look for a difference of squares.
3. If there is a trinomial expression of the form $x^2 + bx + c$, look for factoring by inspection.
4. After factoring, check to see if further factoring is possible.

Polynomial expressions of the form $ax^2 + bx + c$ will be discussed in the next math course.

Factor Pairs.
36
1 36
2 −18
3 12
4 9
6 6



Factor the following.

a) $9x^2 - 36$
$= 9(x^2 - 4)$
$= 9(x-2)(x+2)$

b) $x^2 - 16x - 36$
$= (x+2)(x-18)$

c) $-x^2 + 26x + 27$
$= -1(x^2 - 26x - 27)$
$= -1(x+1)(x-27)$

d) $x^2 - 3x - 5x + 15$
$= x(x-3) - 5(x-3)$
$= (x-3)(x-5)$

Common factor.

Complete Assignment Questions #1 - #9

Assignment

1. Factor.

a) $x^2 - 49$ difference of squares
$= (x-7)(x+7)$

b) $x^2 - 8x + 15$ inspection
$= (x-3)(x-5)$

c) $8x^2 + 32$ common factor
$= 8(x^2 + 4)$

d) $-a^2 + 64$ S1: common factor S2: difference of squares
$= -1(a^2 - 64)$
$= -1(a-8)(a+8)$

e) $e^2 - 3e + 4$ inspection
$= (e-4)(e+1)$

f) $v^2 + 7v + 10$ inspection
$= (v+2)(v+5)$

g) $a^2 + 2ab - 35b^2$ inspection
$= (a-5b)(a+7b)$

h) $4 - 25t^2$ difference of squares
$= (2-5t)(2+5t)$

i) $x^2 + 16$
$=$ not possible

2. Factor.

a) $a^2 - 64b^2$

$= (a-8b)(a+8b)$

b) $108 - 3z^2$

$= 3(36 - z^2)$
$= 3(6-z)(6+z)$

c) $-x^2 - 5x - 4$

$= -1(x^2+5x+4)$
$= -1(x+1)(x+4)$

d) $625p^2 - 1$

$= (25p-1)(25p+1)$

e) $-3x^2 - 3x + 36$

$= -3(x^2+x-12)$
$= -3(x-3)(x+4)$

f) $8v^2 - 32v - 96$

$= 8(v^2-4v-12)$
$= 8(v+2)(v-6)$

3. Factor.

a) $b^2 - 16 - 6b + 24$

$= \underbrace{(b-4)(b+4)} - 6\underbrace{(b-4)}$
$= (b-4)[(b+4)-6]$
$= (b-4)(b-2)$

b) $x^3 - 81x$

$= x(x^2-81)$
$= x(x-9)(x+9)$

can change order instead of removing a negative through factoring.

c) $-256 + t^2$

$= t^2 - 256$
$= (t-16)(t+16)$

d) $12 - 4x - x^2$

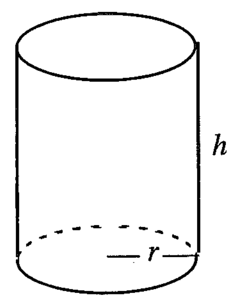
$= -1(x^2+4x-12)$
$= -1(x-2)(x+6)$

Changed order to help see inspection easier.

e) $x^2 - 8xy - 33y^2$

$= (x+3y)(x-11y)$

4. The surface area of a cylinder is given by the formula $A = 2\pi r^2 + 2\pi rh$, where r is the radius of the base and h is the height of the cylinder.



a) Calculate the surface area, to the nearest $0.01\ cm^2$, of a cylinder which has vertical height 14.5 cm and base diameter 11 cm.

$A = 2\pi(5.5)^2 + 2\pi(5.5)(14.5)$
$= 691.15\ cm^2$

remember to divide by 2.

$r = \frac{11}{2} = 5.5$ cm

b) Write the formula for A in factored form.

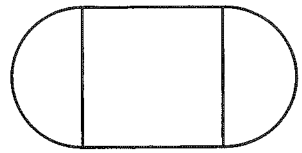
$A = 2\pi r(r+h)$

c) Calculate, using the factored form of A, the surface area of the cylinder to the nearest $0.01\ cm^2$.

$A = 2\pi(5.5)(5.5 + 14.5)$
$= 691.15\ cm^2$

d) Which method a) or c) is simpler to use? c)

5. A square of side $2r$ cm has semicircles drawn externally on each of two opposite sides.



Find expressions in factored form for

a) the external perimeter of the shape

The perimeter consists of the circumference of a circle of radius r and two straight lines each of length 2r.

P =  + 2

$= 2\pi r + 2(2r)$

$\underline{\underline{P = 2r(\pi + 2) \text{ cm}}}$

b) the area of the shape

The area consists of the area of a circle of radius r and the area of a square of side 2r.

A =  + 

$= \pi r^2 + (2r)^2$

$= \pi r^2 + 4r^2$

$A = \underline{\underline{r^2(\pi + 4) \text{ cm}^2}}$

Multiple Choice

Use the following information to answer the next two questions.

In questions #6 -#7 one or more of the four responses may be correct. Answer

A. if only 1 and 2 are correct
B. if only 1, 2, and 3 are correct
C. if only 3 and 4 are correct
D. if some other response or combination of responses is correct

Recall: Always check for common factor first!

6. The set of factors of $5x^2 - 10x - 15$ contains $5(x^2 - 2x - 3) = 5(x+1)(x-3)$

C

1. $x - 1$ **2.** $x + 3$ **3.** $x + 1$ **4.** $x - 3$

7. $x + 4$ is a factor of

B

1. $-x^2 - 6x - 8$
$= -1(x^2 + 6x + 8)$
$= -1(x+2)(x+4)$

2. $48 - 3x^2$
$= -3(x^2 - 16)$
$= -3(x-4)(x+4)$

3. $3x^2 + 12x$
$= 3x(x+4)$

4. $x^2 + 16$
= not factorable.

8. $\pi r^3 + 3\pi r$ is equivalent to

common factor is πr

A. $3\pi^2 r^4$
B. $3\pi(r^2 + r)$
C. $\pi r(2r + 3)$
D. $\pi r(r^2 + 3)$

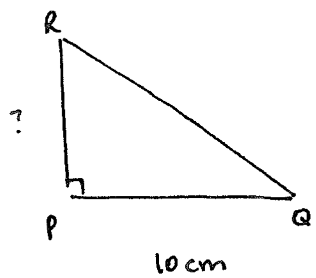
$\pi r(r^3 + 3)$

Numerical Response

9. Triangle PQR is right angled at P. The area of the triangle is $\frac{1}{2}x^2 + 10x + 18$ cm^2, where x is a positive integer.

If the length of PQ is 10 cm, then the length of PR, is ______ cm.
(Record your answer in the numerical response box from left to right)

2	6		



$A = \frac{1}{2}bh = \frac{1}{2}x^2 + 10x + 18$

$\frac{1}{2}bh = \frac{1}{2}(x^2 + 20x + 36)$

$bh = x^2 + 20x + 36$

$\boxed{bh = (x+2)(x+18)}$

Factor Pairs

36

1	36
2	18 = 20
3	12
4	9
6	6

So $x + 2 = 10$

$\boxed{x = 8}$

thus, $x + 18 = (8) + 18 = 26$ The length of PR is <u>26 cm</u>

Answer Key

1. a) $(x-7)(x+7)$ b) $(x-5)(x-3)$ c) $8(x^2+4)$
d) $-(a+8)(x-8)$ e) not factorable f) $(v+5)(v+2)$
g) $(a+7b)(a-5b)$ h) $(2-5t)(2+5t)$ i) not factorable

2. a) $(a-8b)(a+8b)$ b) $3(6-z)(6+z)$ c) $-(x+4)(x+1)$
d) $(25p-1)(25p+1)$ e) $-3(x-3)(x+4)$ f) $8(v+2)(v-6)$

3. a) $(b-2)(b-4)$ b) $x(x-3)(x+3)$ c) $(t-4)(t+4)$
d) $-(x-2)(x+6)$ e) $(x-11y)(x+3y)$

4. a) 691.15 cm^2 b) $A = 2\pi r(r+h)$ c) 691.15 cm^2 d) c) is simpler

5. a) $2r(\pi+2)$ cm. b) $r^2(\pi+4)$ cm^2.

6. C 7. B 8. D 9.

2	6		

Factoring Polynomial Expressions Lesson #6
Enrichment Lesson - Solving Polynomial Equations

Investigation

The **Zero Product Property** states the following:

If $ab = 0$, then either $a = 0$ or $b = 0$

Complete the following to investigate the use of the Zero Product Property (also referred to as the **Zero Product Rule**) in solving polynomial equations.

Isolate x:
$x - 3 = 0$
$+3 \quad +3$
$\boxed{x = 3}$

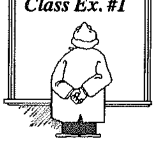
- The statement $x - 3 = 0$ is true only if $x =$ 3.
- The statement $x + 1 = 0$ is true only if $x =$ -1.
- The statement $(x-3)(x+1) = 0$ is true if $x =$ 3 or if $x =$ -1.
- The statement $4(x-3)(x+1) = 0$ is true if $x = 3$ or $x = -1$.

All the above statements are polynomial equations in which the left side is a polynomial expression and the right side equals zero.

The **solution** to a polynomial equation is given by stating the value(s) of the variable which make(s) the left side and the right side equal. These values are said to **satisfy** the equation.

Solving Polynomial Equations

Consider the equation $x^2 - 2x - 3 = 0$. Factoring the left side leads to $(x-3)(x+1) = 0$. This is true if $x = 3$ or if $x = -1$. Since the equation is satisfied by both $x = 3$ and $x = -1$, the solutions to the equation are $x = 3$ **and** $x = -1$, sometimes written as $x = -1, 3$.



Class Ex. #1

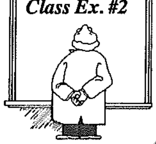
Complete the solution to the equation $x^2 - 9x + 20 = 0$.

$x^2 - 9x + 20 = 0$

$(x - 4)(x - 5) = 0$

$x - 4 = 0$ or $x - 5 = 0$ — The solutions are $x =$ 4 and $x =$ 5

$x =$ 4 or $x =$ 5 — or $x =$ 4, 5



Class Ex. #2

Solve the equation.

a) $x^2 - 81 = 0$ **b)** $4x^2 - 9 = 0$ **c)** $10x^2 - 90x = 0$ **d)** $10x^2 - 90 = 0$.

a) $(x-9)(x+9) = 0$

$x = 9$ or $x = -9$

$x = \pm 9$

b) $(2x-3)(2x+3) = 0$

$x = \frac{3}{2}$ or $x = -\frac{3}{2}$

$x = \pm\frac{3}{2}$

c) $10x(x-9) = 0$

$x = 0$ or $x = 9$

$x = 0, 9$

d) $10(x^2 - 9) = 0$

$10(x-3)(x+3) = 0$

$x = 3$ or $x = -3$

$x = \pm 3$

Set each factor to zero

$x - 9 = 0$
$x = 9$
$x + 9 = 0$
$x = -9$

Class Ex. #3

Solve the following equations.

a) $(3x+2)(x-5)=0$

$x = \frac{-2}{3}$ or $x = 5$

$x = \frac{-2}{3}, 5$

Notice that this polynomial is already factored. We simply need to let each factor equal zero.

$x-5=0$
$+5 \quad +5$
$\boxed{x=5}$

or

$3x+2=0$
$-2 \quad -2$
$\frac{3x}{3} = \frac{-2}{3}$
$\boxed{x=\frac{-2}{3}}$

b) $5x^2+30x=-25$

Problem: This polynomial is not equal to zero. We must add 25 to both sides.

$5x^2+30x+25=0$
$5(x^2+6x+5)=0$
$5(x+1)(x+5)=0$

$\rightarrow$ $x=-5$ or $x=-1$

$x=-5, -1$

Complete Assignment Questions #1 - #3

Problem Solving with Polynomial Equations

Some problems in mathematics can be solved by the following procedure:

i) Introduce a variable to represent an unknown value.

ii) Form a polynomial equation from the given information.

iii) Solve the polynomial equation using the methods in this lesson.

iv) State the solution to the problem.

In this section we will consider fairly routine problems. This topic will be extended in a higher level math course.

Class Ex. #4

The area of a rectangular sheet of paper is 300 cm^2. The length is 5 cm more than the width. Form a polynomial equation and solve it to determine the perimeter of the rectangular sheet.

Step 1: Unknown

length = x cm
width = x − 5 cm

Known.

Area = 300 cm^2

Step 2: Area $\square$ = lw = $x(x-5)=300$
$x^2-5x=300$
$-300 \quad -300$

$x^2-5x-300=0$
$(x-20)(x+15)=0$

$x=20$ or $x=-15$

S3: Check all possible answers: Reject −15 since length cannot be negative.

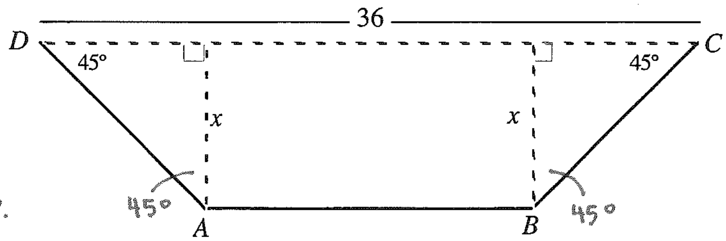
S4: length = x = $\boxed{20\text{ cm}}$

then width = x − 5
= (20 − 5)
= $\boxed{15\text{ cm}}$

S5: perimeter = 2(20) + 2(15)
= 70 cm

Class Ex. #5

The diagram shows the cross-section of a water trough whose sloping sides AD and BC make an angle of $45°$ with the horizontal. The length $DC = 36$ cm.



The two triangles must be isosceles since $180° - 90° - \underline{\underline{45°}} = \underline{\underline{45°}}$.

a) Show that the area of the cross-section is $x(36 - x)$ cm^2.

Recall:
Area of a trapezoid is $A = \frac{1}{2}h(a+b)$

$$\text{Area} = \frac{1}{2} \times \left[(36 - 2x) + 36\right] = \frac{1}{2} \times (72 - 2x) = x(36 - x)\text{cm}^2$$

b) If the area of the cross-section is 260 cm^2, determine the value of x.

Step1: $x(36 - x) = 260$
$36x - x^2 = 260$

Problem: This polynomial is not equal to zero. We must either simplify both terms from the left side to the right side or subtract 260 from each side.

Step2: since the leading coefficient is negative we will begin by adding x^2 to both sides.

$36x - x^2 = 260$
$+x^2 \quad +x^2$

$36x = x^2 + 260$
$-36x \quad -36x$
$0 = x^2 - 36x + 260$

S3: Factor.
$x^2 - 36x + 260 = 0$
$(x - 10)(x - 26) = 0$
$\boxed{x = 10}$ or $\boxed{x = 26}$

S4: Since $AB = 36 - 2x$, x cannot $= 26$.
$AB \neq -16$.

Thus $x = \underline{\underline{10\text{cm}}}$

Complete Assignment Questions #4 - #8

Assignment

1. Solve the equation.

$x - 2 = 0$
$+2 \quad +2$
$\boxed{x = 2}$

$x + 7 = 0$
$-7 \quad -7$
$\boxed{x = -7}$

a) $(x - 2)(x + 7) = 0$

$x = 2$ or $x = -7$

$x = 2, -7$

b) $(3x - 2)(2x + 5) = 0$

$x = \frac{2}{3}$ or $x = -\frac{5}{2}$

$x = \frac{2}{3}, -\frac{5}{2}$

c) $5x(10 - x) = 0$

$x = 0, 10$

d) $x^2 + 2x = 0$

$x(x + 2) = 0$

$x = 0, -2$

e) $x^2 - 121 = 0$

$(x - 11)(x + 11) = 0$

$x = \pm 11$

f) $9x^2 - 100 = 0$

$(3x - 10)(3x + 10) = 0$

$x = \pm\frac{10}{3}$

Recall: Difference of Squares
$\sqrt{49} = 7$
$\sqrt{x^2} = x$

Zero Product Property.

g) $36x^2 = 25$

$36x^2 - 25 = 0$

$(6x - 5)(6x + 5) = 0$

$x = \pm\frac{5}{6}$

h) $9x - 4x^2 = 0$

$x(9 - 4x) = 0$

$x = 0, \frac{9}{4}$

i) $4(49 - x^2) = 0$

$4(7 - x)(7 + x) = 0$

$x = \pm 7$

2. Solve the equation.

a) $x^2 - 3x + 2 = 0$

$(x-1)(x-2)=0$

$x = 1, 2$

b) $x^2 + 13x + 30 = 0$

Recall: Factor Pair 30

1 30
2 15
3 10 → 3 + 10 = 13
5 6
~~6 5~~

Stop

$(x+10)(x+3) = 0$

$x = -10, -3$

c) $x^2 + 2x - 15 = 0$

$(x+5)(x-3)=0$

$x = -5, 3$

3. Solve the equation.

a) $x(x+4) = 32$

Step 1: Expand and simplify, moving all terms to the Left hand side.

$x^2 + 4x = 32$
$-32 \quad -32$

S2: $x^2 + 4x - 32 = 0$

Zero Product Rule.

$(x+8)(x-4)=0$

$x = -8, 4$

b) $(2x-3)^2 = 1$

$(2x-3)(2x-3) = 1$
$4x^2 - 12x + 9 = 1$
$4x^2 - 12x + 8 = 0$
$4(x^2 - 3x + 2) = 0$
$4(x-1)(x-2) = 0$

$x = 1, 2$

c) $(x+1)(x-1) = 5(x+1)$

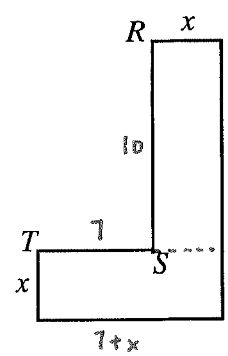
Expanding a difference of squares.

$x^2 - 1 = 5x + 5$

Apply the Zero Product Rule and make the Right Hand Side equal to zero.

$x^2 - 5x - 6 = 0$
$(x+1)(x-6) = 0$
$x = -1, 6$

4. The diagram shows a piece of wood of uniform width x cm. $RS = 10$ cm and $ST = 7$ cm.



a) Find the area of the piece of wood in terms of x.

Area = ▯ + ▭

$= 10x + x(7+x)$
$= 10x + 7x + x^2 \quad = \underline{\underline{17x + x^2}}\ \text{cm}^2$

b) Find the value of x if the area is $60\ \text{cm}^2$.

$17x + x^2 = 60$ ← Apply the Zero Product Rule.
$-60 \quad -60$

$x^2 + 17x - 60 = 0$
$(x+20)(x-3) = 0$
$x = -20, 3$

reject $x = -20$ since a side must have a positive value $(x > 0)$

Thus, $\underline{\underline{x = 3}}$

5. The sum of the first n even numbers, starting with 0, is given by the formula $S = n(n-1)$.

a) Determine the sum of the first 25 even numbers, starting with 0.

Sum $= 25(25-1) = 600$

b) How many consecutive even numbers, starting with 0, add up to 870?

Apply the Zero Product Rule. → $n(n-1) = 870$

$n^2 - n - 870 = 0$
$(n-30)(n+29) = 0$

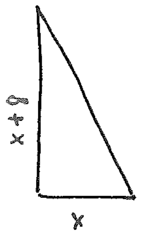
$n = 30, -29$

Reject $n = -29$ since -29 cannot be a count number of how many consecutive even numbers there are that add up to 870.

$n = 30$ $\quad$ $\underline{\underline{30}}$ consecutive even numbers.

6. The height of a triangle is 8 mm more than the base. The area is 172.5 mm^2.

a) Write a polynomial equation to model this information.



Let base $= x$ mm
height $= x + 8$ mm

Recall: Area $= \frac{1}{2}(b \cdot h)$

$A = \frac{1}{2}x(x+8) = 172.5$

$x(x+8) = 345$

$\underline{x^2 + 8x - 345 = 0}$

b) Determine the height of the triangle.

$x^2 + 8x - 345 = 0$

$(x - 15)(x + 23) = 0$

$x = 15, -23$

factor pairs
345
$-1 \quad +345$
$-5 \quad +69$
$-15 \quad +23 = 8$
$\cancel{23 \quad 15}$ ← STOP

Let $x = 15$
Thus, $x + 8 = 15 + 8 = 23$

So the height is $\underline{23 \text{ mm}}$

reject $x = -23$ since a side length must be greater than zero.
$x > 0$

Multiple Choice

7. The complete solution to the equation $x(x - 1) = 2$ is

A. $x = 0$ and $x = 1$
B. $x = 2$ and $x = 3$
C. $x = -1$ and $x = 2$
D. $x = -2$ and $x = 1$

Step 1: Apply the zero Product Rule, making the right side equal to zero.
$x^2 - x - 2 = 0$
Step 2: Factor and isolate for x.
$(x+1)(x-2) = 0$
$x = -1, 2$

Numerical Response

8. The sum of the first n natural numbers is given by the formula $S = \frac{1}{2}n(n + 1)$. If the first k natural numbers have a sum of 496, the value of k is _____.

(Record your answer in the numerical response box from left to right)

3	1		

$\frac{1}{2}k(k+1) = 496$ ← multiply both sides by 2. to begin applying the zero Product Rule.

$k(k+1) = 992$

$k^2 + k = 992$ ← subtract 992 from both sides of the equation.

$k^2 + k - 992 = 0$

$(k-31)(k+32) = 0$ ← Factor using inspection.

$k = 31, -32$

reject -32 since $k > 0$.

Thus, $\underline{k = 31}$

Answer Key

1. **a)** $2, -7$ **b)** $\frac{2}{3}, -\frac{5}{2}$ **c)** $0, 10$ **d)** $0, -2$
e) ± 11 **f)** $\pm\frac{10}{3}$ **g)** $\pm\frac{5}{6}$ **h)** $0, \frac{9}{4}$ **i)** ± 7

2. **a)** $1, 2$ **b)** $-10, -3$ **c)** $-5, 3$

3. **a)** $-8, 4$ **b)** $1, 2$ **c)** $-1, 6$

4. **a)** $x^2 + 17x$ cm^2 **b)** 3 **5.** **a)** 600 **b)** 30

6. **a)** $x^2 + 8x - 345 = 0$ **b)** 23 mm **7.** C **8.**

3	1		

Factoring Polynomial Expressions Lesson #7:
Practice Test

lowest common exponent.

1. One factor of $9x^4 - 6x^3 + 3x^2$ is

9 → 3, 3   6 → 2, 3   3

A. $9x^4$
B. $3x^2 - 2x$ $3x^2(3x^2 - 2x + 1)$
C. $3x^2 - 6x + 3$
D. $3x^2 - 2x + 1$

GCF = 3

2. When fully factored, the expression $x^3y^2 - x^2y^3$ is written

A. $xy^2(x - xy)$ $x^2y^2(x - y)$
B. $x^2y(xy - y)$
C. $x^2y^2(x - y)$
D. $x^3y^2(1 - xy)$

Numerical Response

1. When the greatest common factor is removed from the binomial $75x - 50x^2$, the binomial can be written in the form $ax(b + cx)$. The value of $a - b - c$ is _____ .

(Record your answer in the numerical response box from left to right)

2	4		

75 → 3, 25 → 5, 5   50 → 2, 25 → 5, 5

$50 = 2 \times 5 \times 5$
$75 = 3 \times 5 \times 5$

GCF = 25

$25x(3 - 2x)$ → a, b, c

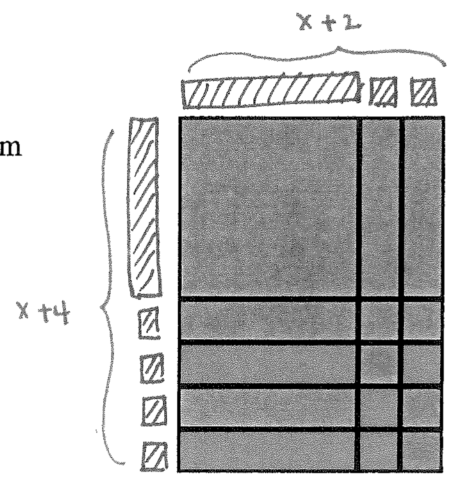
$a - b - c = 25 - 3 - (-2)$
$= 24$

3. The expression $3ab^2 - 6a^3b + 3ab$, when fully factored, is written

$3ab(b - 2a^2 + 1)$

A. $ab(3b - 2a^2 + 3)$
B. $3a(b^2 - 2a^2b + b)$
C. $3ab(b - 2a^2)$
D. none of these

4. The algebra tile diagram represents the factored form



A. $(x^2 + 2x)(x^2 + 4x)$
B. $8(x + 2)(x + 4)$
C. $(x + 2)(x + 4)$
D. $(x + 1)(x + 8)$

(−)(+) → larger

5. One factor of $a^2 - 10a - 24$ is

$= (a+2)(a-12)$

Factor Pairs
24
+1 −24
+2 −12 = −10
+3 −8
+4 −6

A. $a - 2$
B. $a - 4$
C. $a - 6$
D. $a - 12$

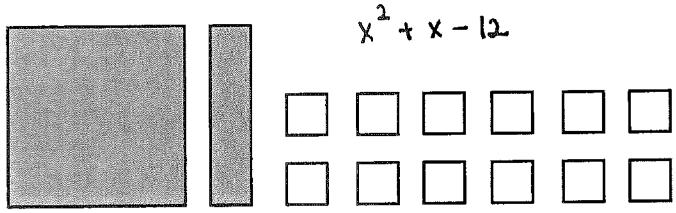
~~6~~4 ← STOP

Use the following information to answer the next question.

An algebraic expression is represented by the algebra tiles shown.

Shaded tiles are positive.

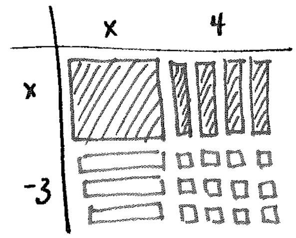
$x^2 + x - 12$



6. The factored form of the algebraic expression represented by the algebra tiles is

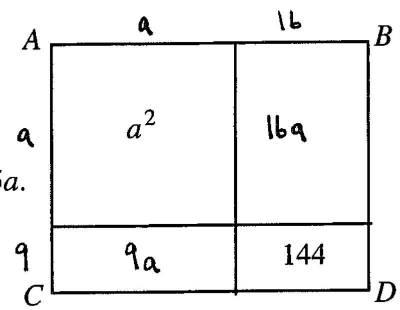
A. $(x-3)(x+4)$
B. $(x-3)(x-4)$
C. $(x-4)(x+3)$
D. x^2+x-12

x^2+x-12
$=(x+4)(x-3)$
So we can also see this factorization using algebra tiles. We just need to include ▨▨▨ and □□□.



Use the following information to answer the next question.

Rectangle *ABCD* has been subdivided into four regions. The areas of two of these regions are a^2 and 144 as indicated. The combined area of the other two regions is $25a$.



find two numbers which have a sum of 25 and a product of 144.

7. The perimeter of rectangle *ABCD* is

A. $a^2 + 25a + 144$
B. $4a + 48$
C. $4a + 50$
D. unable to be determined from the given information

perimeter $= 2(a+16)+2(a+9)$
$= 2a+32+2a+18$
$= 4a+50$

8. For which of the following trinomials is $b+3$ **not** a factor?

A. b^2+3b $= b(b+3)$ ✓ ← common factor
B. b^3-9b $= b(b^2-9) = b(b-3)(b+3)$ ✓ ← common factor and difference of squares.
C. $b^2+2b-15$ $= (b-3)(b+5)$ ✗ ← inspection
D. $b^2-6b-27$ $= (b+3)(b-9)$ ✓ ← inspection.

9. The expression $x^2 + px + 20$ **cannot** be factored over the integers if p has the value

A. -9 $\quad x^2 - 9x + 20$
B. -12 $\quad x^2 - 12x + 20$
C. 21 $\quad x^2 + 21x + 20$
D. 20 $\quad x^2 + 30x + 20$

$p \neq 20$

factor pairs (pos.)

20

+1	+20	= 21
+2	+10	= 12
+4	+5	= 9
~~5~~	~~4~~	← Stop

factor pairs (neg)

20

−1	−20	= −21
−2	−10	= −12
−4	−5	= −9
~~5~~	~~4~~	

Numerical Response **2.** The largest value of w for which $x^2 - wx + 48$ can be factored over the integers is ____.

(Record your answer in the numerical response box from left to right) | 4 | 9 | | |

We need the two integer factors of 48 with the largest sum.

1 and 48 → sum of 49

$x^2 - 49x + 48$
$= (x-1)(x-48)$

Thus, $w = 49$

factor pairs

48

−1	−48	= −49
−2	−24	
−3	−16	
−4	−12	
−6	−8	
~~8~~	~~6~~	

stop

lowest common exponent.

10. One factor of $7a^2 - 28a^4$ is

A. $1 - 2a$
B. $28a^4$
C. $1 - 4a$
D. $1 - a$

$= 7a^2(1 - 4a^2)$
$= 7a^2(1-2a)(1+2a)$

11. Which of the following is a factor of $y^2 - y - 42$?

A. $y - 6$
B. $y + 7$
C. $y - 2$
D. $y + 6$

$= (x+6)(x-7)$

$(-)(+)$

larger |integer| of factor pair.

factor pairs.

−42

+1	−42
+2	−21
+3	−14
+6	−7
~~7~~	~~6~~ ← STOP

Numerical Response **3.** The polynomial expression $4x^2 + 40x + 100$ can be written in the form $A(x + B)^2$. The value for the product AB is _____.

(Record your answer in the numerical response box from left to right) | 2 | 0 | | |

$4x^2 + 40x + 100$

STEP 1: Check for common factor.
$= 4(x^2 + 10x + 25)$

STEP 2: Factor by inspection where possible.
$= 4(x+5)(x+5)$

STEP 3: Check for difference of squares. (In this case not possible)

S4: → $4x^2 + 40x + 100 = 4(x+5)^2$

A b

$AB = (4)(5) = 20$

STEP 4: Complete the question.

Use the following information to answer the next three questions.

In each of questions #12 - 14 four responses are given.
Answer

- **A.** if response 1 and response 2 only are correct
- **B.** if response 1 and response 3 only are correct
- **C.** if response 2 and response 4 only are correct
- **D.** if no response or some other response or combination of the responses is correct

12. Which of the following are factors of $2t^2 + 4t - 30$?

$= 2(t^2 + 2t - 15)$ ← common factor.
$= 2(t+5)(t-3)$ ← inspection

C

Response 1: $t+3$ ✗
Response 2: $t+5$ ✓
Response 3: $t-5$ ✗
Response 4: $t-3$ ✓

13. $x^2 + 25y^2$ has as a factor

$x^2 + 25y^2$ cannot be factored.

No Factors

D

Response 1 $x - 5y$
Response 2 $x - y$
Response 3 $x + 5y$
Response 4 $x + y$

14. The trinomial $x^2 - 12x + c$ can be factored over the integers if

A

Response 1 $c = 20$ $x^2 - 12x + 20 = (x-2)(x-10)$
Response 2 $c = -28$ $x^2 - 12x - 28 = (x+2)(x-14)$
Response 3 $c = -32$ $x^2 - 12x - 32 =$ not possible.
Response 4 $c = -27$ $x^2 - 12x - 27 =$ not possible.

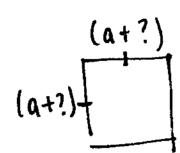
Numerical Response

4. A farmers field has an area of $a^2 + 22a + c$ m^2.
If the field is square, then the value of c must be _______ .
(Record your answer in the numerical response box from left to right)

1	2	1	

$a^2 + 22a + c = (a + \quad)(a + \quad)$

these values must be the same and must add to 22.



$= (a+11)(a+11)$

	a	11
a	a^2	11a
11	11a	121

15. When fully factored, the expression $16a^4 - a^2$ is written

A. $a^2(16a^2 - 1)$

B. $(4a^2 - a)(4a^2 + a)$

C. $(4a^2 - 1)(4a^2 + 1)$

D. $a^2(4a - 1)(4a + 1)$

$= a^2(16a^2 - 1)$ ← common factor

$= a^2(4a-1)(4a+1)$ ← difference of squares.

Numerical Response

5. Three algebraic expressions have been partially factored.

$x^2 - 5x - 14 = (x - 7)(x + A)$

$= (x-7)(x+2)$

	x	-7
x	x^2	-7
A		-14

$(A)(-7) = -14$

So $A = 2$

$x^4 - 9x^2 = x^2(x - B)(x + B)$

$= x^2(x^2 - 9) = (x-3)(x+3)$

$5x^2 - 40x + 80 = C(x - D)^2$

$= 5(x^2 - 8x + 16) = 5(x-4)(x-4) = 5(x-4)^2$

Write the value of A in the first box.
Write the value of B in the second box.
Write the value of C in the third box.
Write the value of D in the fourth box.
(Record your answer in the numerical response box from left to right)

2	3	5	4

Written Response - 5 marks

1. Students are investigating polynomials of the form $x^2 + bx + c$, where b and c are integers.

- State any polynomial with $c = 16$ which can be factored over the integers.

$x^2 + 10x + 16$

- State any polynomial with $c = 16$ which cannot be factored over the integers. **Explain** why factoring is not possible in this case.

$x^2 + 3x + 16$ cannot be factored since it is not possible to find two integers which multiply to 16 and add to 3.

- If $c = 16$, determine how many polynomials of this type can be factored over the integers.

factor pairs

$\pm 1 \quad \pm 16 \quad = \pm 17$ or ± 15

$\pm 2 \quad \pm 8 \quad = \pm 10$ or ± 6

$\pm 4 \quad \pm 4 \quad = \pm 8$ or 0

6 polynomials $x^2 + 10x + 16$ $\quad x^2 + 8x + 16$ $\quad x^2 + 17x + 16$

$x^2 - 10x + 16$ $\quad x^2 - 8x + 16$ $\quad x^2 - 17x + 16$

- State three polynomials of the form $x^2 + bx + c$ which can be factored over the natural numbers and in which $b + c = 19$.

i) All possible values of b and c in the Natural Number System which add up to 19

iii) Use trial and error by replacing the values of b and c in x^2+bx+c. Then use the factor pairs of c to see if 1) $b+c=19$ AND 2) the factor pairs multiply to c and add up to b.

ii)

b	c
1	18
2	17
3	16
4	15
5	14
6	13
7	12
8	11
9	10
10	9

$x^2 + 1x + 18$, Yes because $b+c=19$, $(1)(18) = 18$, and $(b)(c) = 18$.

$x^2 + 2x + 17$, No $b+c=19$, but no two natural factors of 17 multiply to 17 (c) and add up to 2 (b)

$x^2 + 3x + 16$, No $b+c=16$, but no two natural factors of 16 multiply to 16 (c), and add up to 3 (b)

$x^2 + 7x + 12$, Yes $b+c \neq 19$, $(4)(3) = 12$, $4+3=7$

$x^2 + 10x + 9$, Yes $b+c=19$, $(1)(9) = 9$, $1+9=10$

Answer Key

1. D	2. C	3. D	4. C	5. D	6. A	7. C	8. C
9. D	10. A	11. D	12. C	13. D	14. A	15. D	

Numerical Response

1. | 2 | 4 | | |

2. | 4 | 9 | | |

3. | 2 | 0 | | |

4. | 1 | 2 | 1 | |

5. | 2 | 3 | 5 | 4 |

Written Response

1. • $x^2 + 10x + 16$ or any answer in bullet 3.

- Many answers are possible e.g. $x^2 + 3x + 16$ is not able to be factored because it is not possible to find two integers which multiply to 16 and add to 3.

- polynomials are possible $\quad x^2 + 10x + 16 \quad x^2 + 8x + 16 \quad x^2 + 17x + 16$
  $x^2 - 10x + 16 \quad x^2 - 8x + 16 \quad x^2 - 17x + 16$

- The three polynomials are $\quad x^2 + x + 18,\ x^2 + 7x + 12,\ x^2 + 10x + 9$